UNIVERSIDAD DEL CEMA
Buenos Aires
Argentina

Serie
DOCUMENTOS DE TRABAJO

Área: Finanzas

MULTIPLICATIVE MODELS OF FINANCIAL RETURNS AND WHAT WE FAIL TO GET WHEN THEY ARE DISREGARDED

Rodolfo Apreda

Mayo 2011
Nro. 454

ISBN 978-987-1062-65-2
Queda hecho el depósito que marca la Ley 11.723

www.cema.edu.ar/publicaciones/doc_trabajo.html
UCEMA: Av. Córdoba 374, C1054AAP Buenos Aires, Argentina
ISSN 1668-4575 (impreso), ISSN 1668-4583 (en línea)
Editor: Jorge M. Streb; asistente editorial: Valeria Dowding <jae@cema.edu.ar>

Apreda, Rodolfo
Multiplicative models of financial returns and what we fail to get when they are disregarded. - 1a. ed. - Buenos Aires : Universidad del CEMA, 2011.
26 p. ; 22x15 cm.

ISBN 978-987-1062-65-2

1. Finanzas. I. Título.
CDD 332

Fecha de catalogación: 13/06/2011

UNIVERSITY OF CEMA
Working Paper Series

MULTIPLICATIVE MODELS OF FINANCIAL RETURNS

AND WHAT WE FAIL TO GET WHEN THEY ARE DISREGARDED

Rodolfo APREDA

Professor Rodolfo APREDA holds a Ph D. in Economics (University of Buenos Aires) and a Master in Political Sciences (University of Cema). He is director of the Ph D Program in Finance, and of the Center for the Study of Private and Public Governance, both at the University of Cema.
Personal Web Page: www.cema.edu.ar/u/ra E-mail address: ra@cema.edu.ar

Abstract

This paper puts forward an alternative approach to multiplicative models and their assessment of returns out of financial assets. Firstly, it lays down an operative definition but also sets forth a commutative framework of mappings to provide foundations to such a definition. Next, the total return is split down into its linear and non-linear building blocks. Afterwards, a compatibility lemma draws a distinction between what should be meant by linear approximation and linear equivalence to the multiplicative model. Last of all, three empirical examples bring home how to profit from multiplicative models in actual practice.

JEL codes: G11, G12, G17, G30.

Key words*: multiplicative models of returns, additive models of return, financial assets returns, linear approximation and linear equivalences.*

Institutional Disclaimer

Statements or opinions conveyed in this paper are attributable to the author only, and the University of Cema disclaims any responsibility for them.

INTRODUCTION

In the fields of Corporate Finance and Portfolio Management, analysts as well as practitioners usually resort to multiplicative models, although not so often as it would be expected on the grounds of better practices[1].

Roughly speaking, this sort of model assumes that a chosen variable could be explained by the joint effect (or performance) of two or more causative (or explicative) variables, under the guise of an arithmetic product of the latter[2].

We have been concerned with this line of research for the last five years (Apreda, 2010, 2006a). Now and here, our purpose consists in sharpening up the conventional framework of analysis for multiplicative models of return, and setting up a stronger distinction between the linear approximation that comes embedded within any multiplicative model, against the linear equivalence to be derived from the model itself, this topic being rather a recurrent bone of contention among practitioners and academics.

The paper proceeds as follows: in section 1 we bring forwards a basic setting from which our line of argument could move on to next section so as to establish, firstly, what it should be meant by a multiplicative model of financial assets and, secondly, how to factor the model into linear and non-linear parts for the total return. It is for section 3 to lay bare two convertibility lemmas about compatibilities, along with antagonisms, between multiplicative and additive models. Section 4 will focus on the multiplicative model which breaks up inflation from real rates of return, whereas section 5 addresses a multiplicative model that pieces together returns with transaction costs. Section 6 delves into the underlying multiplicative model that deals with uncovered returns from assets held in foreign currencies.

---

[1] Currently available textbooks like those by Ross et al (2009), Damodaran (2006), and Cuthberston (1996) have lately taken heed of multiplicative models.

[2] Section 2 will set forth the definition to be used in this paper.

1. A BASIC SETTING FOR THE ANALYSIS OF MULTIPLICATIVE MODELS

Our starting point of departure will be an investment horizon $\mathbf{H = [\ t;\ T\ ]}$, and the set $\mathbf{U}$ of all financial assets available at a calendar date $\mathbf{t}$, that is to say

$$\mathbf{U} = \{\ \mathbf{A_k} \mid \mathbf{A_k} \text{ is an available financial asset } ;\ \mathbf{k} \in \mathbf{Q} \subset \mathbf{N}\ \}$$

where $\mathbf{Q}$ is a finite index-set out of the set of natural numbers[3].

On the other hand, we are going to denote the set of all likely horizons[4] by means of the following set:

$$\mathbf{Int} = \{\ [\ a;\ b\ ] \subset \mathbf{R}^1 \mid a < b\ \}$$

For each financial asset $\mathbf{A_k}$, we must attach its rate of return along the assumed investment horizon, by eliciting the underlying mapping which works out returns from $\mathbf{U} \times \mathbf{Int}$ upon the set of real numbers $\mathbf{R}^1$:

$$\varphi : \mathbf{U} \times \mathbf{Int} \rightarrow \mathbf{R}^1$$

to be defined by

$$\varphi(\mathbf{A_k};\ [\ t;\ T\ ]) = \mathbf{R}(\mathbf{A_k};\ [\ t;\ T\ ]) = \mathbf{R_k} \tag{1}$$

where[5]

---

[3] If $\mathbf{q = Max\ Q}$, then $\mathbf{Q}$ would be the interval of natural numbers, $\mathbf{Q = \{\ k \in N \mid k = 1, 2, 3, \ldots , q\ \}}$,

[4] About $\mathbf{[\ t;\ T\ ]}$: if $\mathbf{t}$ were actually the starting date of the assessment, and we assimilated it to the ″0-level″ of our analysis, then $\mathbf{[\ t;\ T\ ] = [\ -3;\ 0\ ]}$ would mean we are assessing from three periods before the ″0-level″. Therefore, in what follows, $\mathbf{[\ a;\ b\ ]}$ could carry over one or both extremes negative.

[5] $\mathbf{R(\ A_k\ ;\ [\ t;\ T\ ]\ )}$ is currently called the total return of the financial asset A_k.

$$R(A_k;[t;T]) = \langle V_k(T) - V_k(t) + I_k(t;T) \rangle \,/\, V_k(t)$$

Accordingly, the total return in (1) stems from holding the financial asset from date **t** to date **T** and it embraces changes in value that may take place on both dates, as well as the rewards **I(t; T)** likely to be accrued in the shape of, for instance, interest or dividends.

2. THE MULTIPLICATIVE MODEL

In this section, we are going to move on towards a contextual setting of analysis which intends to answer the following question:

if there were a finite set of rates of change stemming from a set of subsidiary economic variables that explain or are influential to the return $R(A_k;[t;T])$ *of each financial asset, how would each of them relate to the latter?*

To start answering the question raised above, let us assume that there is a set of **Z** economic variables related to the set **U** of available financial assets:

$$[U, Z] = \{X_1, X_2, X_3, \ldots\ldots, X_Z\}$$

whose rates of change[6] lead to the following vector

$$R(U;Z) = [R_1(A_k); R_2(A_k); R_3(A_k); \ldots\ldots; R_Z(A_k)]$$

for any $A_k \in U$. Bearing in mind the former remarks[7], we set about to framing next definition.

[6] Or rates of return, without any loss of generality.

Definition 1

For any given financial asset $A_k \in U$ ***, and any*** $[\,t;\,T\,] \in$ ***Int, we say that the rate of return***

$$R\,(A_k\,;\,[\,t;\,T\,])$$

becomes explained by a multiplicative model coming out of a set of variables $[\,U,\,Z\,]$ ***if the following relationship holds true:***

(2)

$$< 1 + R\,(A_k\,;\,[t;\,T\,]) > \;=$$

$$= < 1 + R_1(A_k\,;\,[t;T\,]) > \times < 1 + R_2(A_k\,;\,[t;\,T\,]) > \times \ldots \times < 1 + R_Z(A_k\,;\,[t;\,T\,]) >$$

Or, equivalently[8]***,***

(3)

$$< 1 + R\,(A_k\,;\,[t;\,T]) > \;=\; \prod_h < 1 + R_h > \quad ;\;(h:\;1, 2, 3, \ldots, Z)$$

We will refer to expression (3) as a multiplicative model, **MM [U, Z]**, for the return of financial assets in **U**, under the explanatory scope of the variables in the set **Z**. Next lemma displays how the multiplicative model defined by (3) can be split up into a linear and a non-linear components.

---

[7] As it is customary, we use square brackets for vectors, curly brackets for sets and angle brackets for distinctive components within certain mathematical relationships. The horizon **[t; T]** will be an exemption from this convention.

[8] For ease of notation, we are going to drop from the right side of expression (3) the symbol standing for the horizon **H**, and also the one for the asset.

Lemma 1

Given an investment horizon $[\,t; T\,] \in Int$, and for any financial asset $A_k \in U$, it holds that the multiplicative model

$$1 + R(k) = \prod_{h} < 1 + R_h > \qquad (h: 1, 2, 3, \ldots, Z)$$

can be factored into the alternative representation

$$1 + R(k) = 1 + \sum_{h(1)} R_{h(1)} + \sum_{h(1) < h(2)} R_{h(1)} \times R_{h(2)} + \sum_{h(1) < h(2) < h(3)} R_{h(1)} \times R_{h(2)} \times R_{h(3)} + \tag{4}$$

$$+ \ldots\ldots\ldots + \sum_{h(1) < h(2) < h(3) < \ldots\ldots < h(Z)} R_{h(1)} \times R_{h(2)} \times R_{h(3)} \times \ldots\ldots \times R_{h(Z)}$$

Proof:

It follows from complete induction that will be developed in Appendix 1. Nonetheless, it′s worth paying notice at this place to the convention we are going to use for indexes:

$h(1)$ is an index that runs from 1 to Z.

$h(2)$ is an index whose values span from $h(1) + 1$ to Z.

By iteration, $h(j)$ is an index whose values span from $h(j) + 1$ to Z. **END OF LEMMA**

Remarks

a) The main outcome from this lemma lies on the fact that we can translate the multiplicative model as coming out of an additive model

(5)

$$\sum \mathbf{R}_{h(j)}$$

and a non-linear expression

$$\Phi\left(\mathbf{R}_1;\ \mathbf{R}_2;\ \mathbf{R}_3;\ \ldots\ldots;\mathbf{R}_N\right) =$$

$$= \sum_{h(1)<h(2)} \mathbf{R}_{h(1)} \times \mathbf{R}_{h(2)} + \sum_{h(1)<h(2)<h(3)} \mathbf{R}_{h(1)} \times \mathbf{R}_{h(2)} \times \mathbf{R}_{h(3)} +$$

$$+ \ldots\ldots\ldots + \sum_{h(1)<h(2)<h(3)<\ \ldots\ldots<h(Z)} \mathbf{R}_{h(1)} \times \mathbf{R}_{h(2)} \times \mathbf{R}_{h(3)} \times \ldots\ldots \times \mathbf{R}_{h(Z)}$$

Hence, expression (4) can be rewritten like

(6)

$$\mathbf{1 + R(k) = 1 +} \sum \mathbf{R}_{h(j)} + \Phi\left(\mathbf{R}_1;\ \mathbf{R}_2;\ \mathbf{R}_3;\ \ldots\ldots;\mathbf{R}_Z\right)$$

b) Still further, we should allow for an alternative environment in which, given any temporal span, and for every financial asset in **U**, the variable $\mathbf{R(A_k; [\ t; T\ ]\ )}$ might be explained by more than one set of subsidiary variables. For instance, we could face two sets of explanatory variables:

$$[\ \mathbf{U, Z}\ ] = \{\ \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \ldots\ldots, \mathbf{X}_Z\ \}$$

$$[\ \mathbf{U, W}\ ] = \{\ \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \ldots\ldots, \mathbf{Y}_W\ \}$$

On this regard, It will be for sections 4, 5, and 6, to match up alternative sets of explanatory variables to empirical settings.

2.1 FORMAL ENLARGEMENT OF THE MULTIPLICATIVE MODEL

Although Definition 1 is fully operational, it might be regarded rather as a pithy statement. However, there are a set of distinctive mappings that provide the scaffolding of the definition. This section will lay bare those mappings so as to lend the definition a more precise mathematical background, while Exhibit 1 will avail the reader of the whole structure of mappings.

a) Firstly, we are going to recall how we had defined in section 1 the mapping that chose both a financial asset $\mathbf{A_k} \in \mathbf{U}$ and a horizon $\mathbf{[\,t;\ T]} \in \mathbf{Int}$, to figure out the financial return of any financial asset along the horizon:

$$\varphi \;:\; \mathbf{U} \times \mathbf{Int} \;\rightarrow\; \mathbf{R}^{1}$$

defined by

$$\varphi\,(\mathbf{A_k}\,;\,[\,\mathbf{t};\ \mathbf{T}\,]) \;=\; \mathbf{R}\,(\mathbf{A_k}\,;\,[\,\mathbf{t};\ \mathbf{T}\,])$$

Afterwards, we define a new mapping φ_1 that will be required for the ensuing argument:

(7)

$$\varphi_1 \;:\; \mathbf{R}^{1} \;\rightarrow\; \mathbf{R}^{1}$$

such that

$$\varphi_1\,(\,\mathbf{R}\,(\mathbf{A_k}\,;\,[\,\mathbf{t};\mathbf{T}\,])\,) \;=\; \mathbf{1} + \mathbf{R}\,(\mathbf{A_k}\,;\,[\,\mathbf{t};\ \mathbf{T}\,])$$

or, simplifying,

$$\varphi_1 \; (R(k)) = 1 + R(k)$$

b) Then, we define a mapping φ_2 that takes **U** $\times$ **Int** into a vector of explanatory economic variables:

(8)

$$\varphi_2 : \mathbf{U} \times \mathbf{Int} \rightarrow \mathbf{R}^Z$$

such that

$$\varphi_2 \; (A_k ; [t; T]) =$$

$$= [X_1(A_k, [t; T]), X_2(A_k, [t; T]), \ldots\ldots, X_Z(A_k, [t; T])]$$

c) From the vector comprising the explanatory variables, we move on to the vector of their corresponding rates of change (or returns):

(9)

$$\varphi_3 : \mathbf{R}^Z \rightarrow \mathbf{R}^Z$$

ruled by

$$\varphi_3 \, [X_1(A_k, [t; T]), X_2(A_k, [t; T]), \ldots\ldots, X_Z(A_k, [t; T])] =$$

$$= [R_1(A_k, [t; T]), R_2(A_k, [t; T]), \ldots\ldots, R_Z(A_k, [t; T])]$$

d) After that, we need a mapping that takes the vector comprising the rates of returns of the explanatory variables into the factorial returns given by relationship (3).

$$\varphi_4 \ : \ R^Z \ \rightarrow \ R^1 \qquad (10)$$

such that

$$\varphi_4 \ [\ R_1(A_k, [\, t; T\,]),\ R_2(A_k, [\, t; T\,]),\ \ldots\ldots,\ R_Z(A_k, [\, t; T\,])\ ] \ =$$

$$= \prod_h < 1 + R_h > \quad ; \ (h: \ 1, 2, 3, \ldots, Z)$$

Exhibit 1 **The commutative diagram brought about by the multiplicative model**

$$\begin{tikzcd}
R^Z \arrow[d, "\varphi_3"'] & U \times Int \arrow[l, "\varphi_2"'] \arrow[d, "\varphi"] \arrow[ddl] \\
R^Z \arrow[d, "\varphi_4"'] & R^1 \arrow[d, "\varphi_1"] \\
R^1 & R^1 \arrow[l, "\varphi_5"']
\end{tikzcd}$$

e) Lastly, we balance up the image of φ_4 with the image of φ_1 by means of a mapping which we are going to denote as φ_5:

(11)

$$\varphi_5 : R^1 \rightarrow R^1$$

such that

$$\varphi_5 \circ (\varphi_1 \circ \varphi) (A_k ; [t; T]) = \varphi_4 \circ (\varphi_3 \circ \varphi_2) (A_k ; [t; T])$$

or, equivalently,

$$1 + R (k) = \prod_h < 1 + R_h > \quad ; (h: 1, 2, 3, \ldots , Z)$$

Therefore, by (11), it holds that

(12)

$$\varphi_5 \circ \varphi_1 \circ \varphi = \varphi_4 \circ \varphi_3 \circ \varphi_2$$

What sort of message does this relationship convey? It tells that the mappings commute when (12) stays true. In such case (see exhibit 1), we say that

$$\varphi_4 \circ \varphi_3 \circ \varphi_2 = \mathbf{MM} [U, Z]$$

that is to say, the multiplicative model **MM [U, Z]** stands for the total return R (k). In other words,

$$\mathbf{MM} [U, Z] = \varphi_5 \circ \varphi_1 \circ \varphi$$

3. TWO CONVERTIBILITY LEMMAS

It goes without saying that in several frameworks of analysis, using the additive model to explain the total return of a financial asset could amount to a hugely farfetched procedure, as the convertibility lemmas 2 and 3 will set forth.

Lemma 2

If $\Phi\left(R_1;\ R_2;\ R_3;\ \ldots\ldots;R_Z\right) = 0$, then the additive model is fully convertible into the multiplicative model.

Proof: it follows directly from (6). **END OF LEMMA**

Remarks

This lemma deserves two comments:

a) As we see, to assume that, in general, $\Phi\left(\mathbf{R}_1;\ \mathbf{R}_2;\ \mathbf{R}_3;\ \ldots\ldots;\mathbf{R}_Z\right) = \mathbf{0}$, it would stand far removed from empirical evidence.

b) However, there should be an empirical yardstick as from which practitioners would be able to approximate the multiplicative model by means of a considerate usage of the additive model. In point of fact, we have lately introduced an alternative yardstick (Apreda, 2010). Bearing in mind that returns and interest rates are customarily formatted with four decimal digits, our metrics happens to be the following[9], which derives from (6)

[9] In fact, 10^{-4} is one basis point (0.0001). The issue here is how much a basis point is worthwhile for the practitioner as a cost of opportunity. If the investment at date **t** amounted to a billion dollars, to disregard a basis point means that the approximation loses 100,000 dollars. Instead, if the yardstick were set up in the order of 10 basis points (10^{-3}), the approximation would be worth up to 1 million dollars.

$$| \mathbf{R(k)} - \sum \mathbf{R}_{h(1)} | = | \Phi (\mathbf{R}_1 ; \mathbf{R}_2 ; \mathbf{R}_3 ; \ldots\ldots ; \mathbf{R}_Z) | < \mathbf{10}^{-4} \tag{13}$$

END OF REMARKS

Sometimes it is read that by means of a useful device any multiplicative model could be translated by an additive one eventually. Albeit next lemma proves that this is attainable, it would amount to a mistake, however, to regard such statement as saying that we can substitute the additive model in (5)

$$\sum \mathbf{R}_{h(j)}$$

for the multiplicative model conveyed by (6). Nevertheless, this misplaced substitution turns out to be a widespread usage among many practitioners. Let us cope with this issue in further detail.

Lemma 3

Assuming a continuous generating process of returns, the multiplicative model can be translated into an additive model.

Proof: recalling the expression (3) of the multiplicative model for the return of any financial asset

$$< \mathbf{1} + \mathbf{R} (\mathbf{A}_k ; [\mathbf{t} ; \mathbf{T}]) > = \prod_{g} < \mathbf{1} + \mathbf{R}_g > \quad ; (\mathbf{g} : \mathbf{1, 2, 3, \ldots , Z})$$

and assuming a continuous generating process[10] we get

$$< 1 + R(A_k; [t; T]) > = e^{\lambda(k)(T-t)}$$

$$\prod < 1 + R_g > = \prod e^{\lambda(g)(T-t)}$$

by taking logarithms

$$\ln < 1 + R(A_k; [t; T]) > = \ln e^{\lambda(k)(T-t)}$$

$$\ln \prod < 1 + R_g > = \ln \prod e^{\lambda(g)(T-t)}$$

which leads to

$$\ln < 1 + R(A_k; [t; T]) > = \lambda(k)(T-t) \tag{14}$$

$$\ln \prod < 1 + R_g > = \sum \lambda(g)(T-t) \tag{15}$$

plugging (14) and (15) into (3), and dropping out (**T – t**) we get

$$\lambda(k) = \sum \lambda(g) \tag{16}$$

[10] In a continuous process like this one, **V(t)** accrues to **V(T) + I(t, T)**, by means of an instantaneous rate of return **λ(.)**. Therefore:

$$< < V(T) + I(t, T) > / V(t) > = 1 + R(k) = e^{\lambda(\cdot)(T-t)}$$

Hence, departing from the multiplicative model in (3), we arrive at a translation of it into an additive expression. **END OF LEMMA**

Remarks

a) Some people stand up for additive models like (5) on the grounds of their simplicity. Whenever such approximation were tenable, nothing could be more up to the mark.

b) At the end of the day, if we wanted a linear approximation to the multiplicative model, we should seek for

$$\sum \mathbf{R}_{h(j)}$$

but if we sought a linear equivalence to the multiplicative model instead, we should have dealt with (16):

$$\lambda(\mathbf{k}) = \sum \lambda(\mathbf{g})$$

c) At last, but not least, expression (6) conveys essentially good practical value, since it is the natural way of working out the real rate of return of financial assets and portfolios of financial assets. In contrast, the linear equivalence of lemma 3 lacks of intuitive appeal and does not bring to light the returns of the explicative variables. The trouble with the linear equivalence is not that it is wrong, but rather it is uninformative, by which (6) comes as a bonus eventually. **END OF REMARKS**

4. INFLATION AND REAL RETURNS

We can find a crystal clear example of a multiplicative model when we attempt to relate the nominal rate of return with the real rate of return. The following relationship is a time-honored proposition by Fisher(1898), predicated upon rates of interest within money markets, but easily enlarged to hold in capital markets as well.

Let us denote by

$$\mathbf{real\ (\ k\ )}$$

the real rate of return of a financial asset $\mathbf{A_k} \in \mathbf{U}$, over the whole investment horizon $\mathbf{H = [\ t;\ T\ ]}$, and by $\boldsymbol{\pi = \pi(t;\ T)}$ the expected rate of inflation (or realized, if we evaluated at date **T**).

Lemma 4

For any financial asset $A_k \in U$ and any $[\ t;\ T\ ] \in Int$, it holds that

$$\boldsymbol{1 + R(\ k\ ) = < 1 + \pi > \times < 1 + real\ (\ k\ ) >}$$

Proof: being the real rate of return that return stemming from values adjusted by inflation, we have:

$$\mathbf{1 + real\ (\ k\ ) = < V(\ T\ ) + I(\ t;\ T\ ) > \ / \ (\ V(\ t\ ) \times < 1 + \pi >)} \tag{17}$$

which leads to

$$\mathbf{< V(\ T\ ) + I(\ t;\ T\ ) \ / \ V(\ t\ ) > \ = \ < 1 + \pi > \times < 1 + real\ (\ k\ ) >}$$

Or, equivalently,

$$1 + R(k) = \langle 1 + \pi \rangle \times \langle 1 + \text{real}(k) \rangle$$ **END OF LEMMA**

Remarks

a) The basic consequence of this lemma amounts to positively answering the question: Does a multiplicative model actually exist down to earth?

b) There is a stronger connection between multiplicative models, differential rates of returns and residual information sets, which we have dealt with elsewhere (Apreda, 2006a, 2004, 2001a, 2001b, 2000).

c) Let us map the commutative structure of mappings displayed in exhibit 1 onto the scaffolding of lemma 1.

- *Mapping* φ $\qquad \varphi(A_k, [t; T]) = R(A_k, [t; T]) = R(k)$

- *Mapping* φ_1 $\qquad \varphi_1(R(k)) = 1 + R(k)$

- *Mapping* φ_2 $\qquad \varphi_2(A_k, [t; T]) = [P(t); 1 + \text{real}(k)]$

where **real (k)** comes defined out of (17) as

$$\text{real}(k) = R(k) / (1 + \pi)$$

- *Mapping* φ_3

$$\varphi_3([P(t); 1 + \text{real}(k)]) = [P(T) / P(t); 1 + \text{real}(k)]$$

- *Mapping* φ_4

$$\varphi_4([P(T)/P(t);\, 1+ \text{real}(k)]) = \langle 1+\pi \rangle \times \langle 1+ \text{real}(k) \rangle$$

- *Mapping* φ_5

$$\varphi_5(1 + R(k)) = 1 + R(k) = \langle 1+\pi \rangle \times \langle 1+ \text{real}(k) \rangle$$

END OF REMARKS

5. TRANSACTION COSTS

For all intents and purposes, transaction costs are to be reckoned with. In point of fact, they matter[11] in any down-to-earth application. Sometimes, one still hears that they are negligible, or even that they have been decreasing through technological advances, but both statements are far from being tenable. In point of fact, there are five groups of transaction costs[12]:

- Trading (tra)
- Information (inf)
- Taxes (tax)
- Financial costs related to the single transaction (fin)
- Microstructure costs (mst)

It would be convenient to embody transaction costs into a multiplicative model of financial assets returns. But for the sake of a stronger semantics, let us move on through the building up of a multiplicative model for transaction costs firstly, leaving their full embedding into the returns model for the following lemma.

(18)

[11] On this topic, both Damodaran (1997) and Spulberg (1996) are worthy of being read.

[12] See remark b), section 4.

$$1 + trc(k) = \langle 1 + tra(k) \rangle \times \langle 1 + inf(k) \rangle \times$$

$$\times \langle 1 + fin(k) \rangle \times \langle 1 + tax(k) \rangle \times \langle 1 + mst(k) \rangle$$

Lemma 5

For any financial asset $A_k \in U$ and any $[t; T] \in Int$, it holds that

$$1 + R(k) = \langle 1 + trc(k) \rangle \times \langle 1 + R(k; \text{net of } trc\ (k)) \rangle \tag{19}$$

Proof: it follows from (18) and defining

$$R(k; \text{net of } trc\ (k))$$

as the differential rate of return that hinges upon ***R(k)*** and ***trc(k)***. **END OF LEMMA**

6. UNCOVERED RETURN IN DEALING WITH FOREIGN ASSETS

Let us assume that we have to handle a topical decision-making procedure in global markets, consisting in the purchase of a financial asset either in a domestic exchange, to be denoted by **DOM**, or a foreign one[13], to be denoted by **FOR**. If the domestic financial asset return is R_{DOM} and its foreign counterpart R_{FOR}, decision-making will be to purchase the former only when its return surpasses not only the latter return but the swap return

$$1 + R_{SWAP}(t; T) = FX^{b}(T) / FX^{a}(t) \tag{20}$$

[13] Further background on this matter in Apreda (2006b).

arising out of buying each unit of foreign currency at date **t** at a price of $\mathbf{FX^{a}(t)}$ in domestic currency, just to be sold later at date **T** at a price of $\mathbf{FX^{b}(T)}$ in domestic currency.

When the setting above-mentioned it does not hold, we will either invest in the foreign exchange, or it would the case that both exchanges are extremely arbitraged in which case we look for other alternative investments. This gives rise to a multiplicative model to explain the uncovered arbitrage in dealing with foreign assets.

Lemma 6

The uncovered arbitrage between domestic and foreign exchanges is explained by the following multiplicative model

(21)

$$1 + R_{DOM}(t;T) =$$

$$= \langle 1 + R_{FOR}(t;T) \rangle \times \langle 1 + R_{SWAP}(t;T) \rangle \times \langle 1 + g(t;T) \rangle$$

where g (t; T) stands for the arbitrage gains (or losses) to be taken if we purchased the domestic financial asset.

Proof: the total return from purchasing at date **t** a financial asset $\mathbf{A_k}$ in the domestic exchange **DOM** will give a total investment value at date **T** equal to [14]

$$1 + R_{DOM}(t;T)$$

[14] If we assessed the return at date **t**, we should use expected values, whereas if we did so at date **T**, we should use realized values. However, we drop the expected operator on behalf of simplicity and without losing generality.

whereas the total return from purchasing at date **t** a financial asset $\mathbf{A}_{\mathbf{j}}$ in the foreign exchange[15] will give a total investment value at date **T** equal to

$$< \mathbf{1 + R_{FOR}(t; T)} >$$

Bringing the two settings into comparable terms, and choosing the domestic exchange as the actual center point for trading, we need to introduce the return, positive or negative, which foreign exchange will bring about by perfecting both transactions of purchasing and selling the foreign exchange, that is to say, the swap return defined by (20)

$$< \mathbf{1 + R_{SWAP}(t; T)} >$$

So, the domestic investment will be the best choice whenever

$$\mathbf{1 + R_{DOM}(t; T)} \; > \; \mathbf{(1 + R_{FOR}(t; T))} \times \mathbf{(1 + R_{SWAP}(t; T))} \tag{22}$$

whereas the foreign investment will more profitable if

$$\mathbf{1 + R_{DOM}(t; T)} \; < \; \mathbf{(1 + R_{FOR}(t; T))} \times \mathbf{(1 + R_{SWAP}(t; T))} \tag{23}$$

Now we introduce the return that closes the gap between (22) and (23). It will be denoted **g (t; T)**. It will carry the gains from purchasing the financial asset in the domestic exchange (if positive) and the losses to be incurred if rejecting the financial asset in the foreign exchange (if negative). Therefore, the following multiplicative model holds true:

$$\mathbf{1 + R_{DOM}(t; T)} \; =$$

[15] Notice that $\mathbf{A}_{\mathbf{j}}$ could be the same $\mathbf{A}_{\mathbf{k}}$ eventually.

$$= <1 + R_{FOR}(t;T)> \times < 1 + R_{SWAP}(t;T)> \times < 1 + g(t;T)>$$ **END OF LEMMA**

CONCLUSIONS

Firstly, the paper has deployed an alternative setting for the analysis of multiplicative models of financial returns, while it has called for a commutative structure of mappings that lay foundations to those multiplicative models.

Secondly, it has brought to light a decomposition of the total return into two components, a linear approximation, and a non-linear approximation.

Next, it has established a pragmatic metrics of acceptance between the linear approximation and the total return itself.

Afterwards, it made a distinctive precision on the antagonisms between the linear approximation and the linear equivalence to the multiplicative model.

Lastly, and for the sake of illustration, it provided with three factual settings that give empirical grounds to the former analysis.

REFERENCES

Apreda, R. (2010) *Cost of Capital Adjusted for Governance Risk through a Multiplicative Model of Expected Returns*. Risk Governance and Control: Market and Institutions, volume 1, number 1.
(downloadable from www.cema.edu.ar/publicaciones and www.ssrn.org)

Apreda, R. (2006a) *Differential Rates, Residual Information Sets and Transactional Algebras*. Nova Science Publishers, New York.

Apreda, R. (2006b) *Arbitrage in Foreign Exchange Markets within the Context of a Transactional Algebra.* In “International Trade Issues”, edited by Professor Robert Weeks, Nova Publishers, New York.
(downloadable from the author’s web page: www.cema.edu.ar/u/ra)

Apreda, R. (2004) *Differential Rates, Residual Information Sets and Transactional Algebras*. Working Paper Series, number 256, Ucema. (E).
(downloadable from www.cema.edu.ar/publicaciones and www.ssrn.org)

Apreda, R. (2001a) *Differential Rates of Return and Residual Information Sets*, Working Paper Series, S-01-3, New York University, Salomon Center, January.

Apreda, R. (2001b) *The Brokerage of Asymmetric Information*, Working Paper Series, number 190, Ucema. (E)
(downloadable from www.cema.edu.ar/publicaciones and www.ssrn.org)

Apreda R. (2000) *Differential Rates of Return and Residual Information Sets*, Working Paper, N°177, Ucema. (E)
(downloadable from www.cema.edu.ar/publicaciones and www.ssrn.org)

Cuthberston, K. (1996) *Quantitative Financial Economics*. John Wiley, New York.

Damodaran, A. (1997) *The Hidden Costs of Trading.* In “Investment Management”, edited by Peter Bernstein and Aswath Damodaran, John Wiley, New York.
(downloadable from www.jstor.org)

Damodaran, A. (2006) *Damodaran on Valuation: Security Analysis for Investment and Corporate Finance.* John Wiley, New York.

Fisher, I. (1896) *Appreciation and Interest*. American Economic Association, Vol. 11, No. 4 (Jul., 1896), pp. 1-98 (downloadable from www.jstor.org)

Ross, S.; Westerfield, R.; Jaffe, J. (2009) *Corporate Finance*, 9th Edition, McGraw-Hill, New York.

Spulber, D. (1996) *Market Microstructure and Intermediaries.* Journal of Economic Perspectives, vol. 10, pp. 135-152. (downloadable from www.jstor.org)

APPENDIX 1

Proof of Lemma 1 *(the convention over indexes was explained in lemma 1, section 2)*

Assume that the index ceiling is **h = 2.**

$$\mathbf{1 + R = (1 + R_1) \times (1 + R_2)}$$

In such case,

$$1 + R = 1 + R_1 + R_2 + R_1 \times R_2$$

where

$$\Phi(R_1;\ R_2) = R_1 \times R_2$$

Therefore, (2) follows for **h = 2**.

Now, let us assume that (2) holds when **h = Z – 1**. We want to prove that it also holds true when **h = Z.**

From

$$[1 + R] = \prod_{h} [1 + R_h] \qquad (h:\ 1, 2, 3, \dots, Z)$$

we are led to

$$[1 + R] = \left(\prod_{h < Z} [1 + R_h]\right) \times (1 + R_Z)$$

that can be translated as

$$[1 + R] =$$

$$= \left(\prod_{h < Z} [1 + R_h]\right) + \left(\prod_{h < Z} [1 + R_h] \times R_Z\right)$$

On the right-side of last expression, the first term adds up to the inductive hypothesis, hence

$$1 + R(k) = 1 + \sum_{h(1)} R_{h(1)} + \sum_{h(1) < h(2)} R_{h(1)} \times R_{h(2)} + \sum_{h(1) < h(2) < h(3)} R_{h(1)} \times R_{h(2)} \times R_{h(3)} +$$

$$+ \quad \cdots\cdots \quad + \sum_{h(1) < h(2) < h(3) < \;\ldots\ldots < h(Z-1)} R_{h(1)} \times R_{h(2)} \times R_{h(3)} \times \ldots\ldots \times R_{h(Z-1)} \quad +$$

$$+ \left\{ 1 + \sum_{h(1)} R_{h(1)} + \sum_{h(1) < h(2)} R_{h(1)} \times R_{h(2)} + \sum_{h(1) < h(2) < h(3)} R_{h(1)} \times R_{h(2)} \times R_{h(3)} + \right.$$

$$\left. + \quad \cdots\cdots \quad + \sum_{h(1) < h(2) < h(3) < \;\ldots\ldots < h(Z-1)} R_{h(1)} \times R_{h(2)} \times R_{h(3)} \times \ldots\ldots \times R_{h(Z-1)} \right\} \times R_Z$$

and by distributing and rearranging we finally reach to

$$1 + R(k) = 1 + \sum_{h(1)} R_{h(1)} + \sum_{h(1) < h(2)} R_{h(1)} \times R_{h(2)} + \sum_{h(1) < h(2) < h(3)} R_{h(1)} \times R_{h(2)} \times R_{h(3)} +$$

$$+ \quad \cdots\cdots \quad + \sum_{h(1) < h(2) < h(3) < \;\ldots\ldots < h(Z)} R_{h(1)} \times R_{h(2)} \times R_{h(3)} \times \ldots\ldots \times R_{h(Z)}$$

therefore, (3) also holds when **h = Z**, and by the principle of complete induction, Lemma 1 stays proved. **END OF LEMMA**