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DIFFERENTIAL RATES, RESIDUAL INFORMATION SETS AND TRANSACTIONAL ALGEBRAS

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ABSTRACT

The purpose of this paper is to model differential rates over residual information sets, so as to shape transactional algebras into operational grounds. Firstly, simple differential rates over residual information sets are introduced by taking advantage of finite algebras of sets. Secondly, after contextual sets and the relevant algebra of information sets is suitably fashioned, generalized differential rates over residual information sets are expanded on, while a recursive algorithm is set forth to characterize such rates and sets. Thirdly, the notion of transactional algebra is presented and heed is given to the costs of running such structure. Finally, an application to financial arbitrage processes is fully developed within a transactional algebra, setting up arbitrage returns net of transaction costs, establishing boundary conditions for an arbitrage to take place, and finally allowing for a definition of what should be meant by financial arbitrage within a transactional algebra.

JEL: G10, G12, G14

Key words: differential rates, residual information sets, transactional algebras, arbitrage

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INTRODUCTION

This paper brings to an end a line of research started three years ago when enjoying a Visiting Fellow appointment at the Stern School of Business, New York University. Its expansion will take advantage of former papers by the author (Apreda, 2001a, 2001b). To expand on the main content of this paper, we are going to draw heavily from our two contributions to the subject, that were published in the Working Paper Series, University of Cema: On the Extent of Arbitrage Constraints within Transactional Algebras (2003), and Differential Rates of Return and Residual Information Sets (2000).

In section 1, an outline of information sets is provided. Next, in section 2, finite algebras of sets are introduced. Afterwards, section 3 shapes the notion of simple differential rates over residual information sets. It is for section 4 to make explicit contextual sets and the relevant algebra of information sets. Then, in section 5, a lemma gives an inductive construction of generalized differential rates over residual information sets. Section 6 expands on transactional algebras and the costs of running such structure. Finally, section 7 works out the notion of financial arbitrage within transactional algebras, establishing both existence and boundary conditions for an arbitrage to be successful in such context. An appendix sets forth the recursive program assumed in section 5.

1. INFORMATION SETS

The scaled change of securities prices in a period becomes its total rate of return by means of the relation

$$r(t,T) = [\,P(T) + I(t,T) - P(t)\,] \;/\; P(t) \tag{1}$$

where **P(t)** and **P(T)** stand for prices at the beginning and the end of the holding period, whereas **I(t,T)** stands for cash inflows provided by the security along the holding period (like dividends or interest). Most of the time, we don't know the value of either **P(T)** or **I(t, T)** at date " **t** ". On an ex~ante basis we can substitute estimated values for **P(T)** and **I(t , T)**, namely :

$$E[\,P(T)\,] + E[\,I(t,T)\,]$$

This last value is conditional on the economic agent's "information set" Ω_t . ***An information set means the set of all available information to him, up to the valuation date***.

Remark

It is by no means assumed here symmetric information, and even less that all the information from Ω_t be factored into prices costlessly and instantaneously, an extreme environment claimed by the Efficient Market theory. Therefore, there is enough latitude to encompass more realistic settings, although keeping the ideal one fully operational as needed.

Therefore, rates of change carry on a conditional feature upon future states of the world.

$$<E[\,P(T), \Omega_t\,] + E[\,I(t,T,\Omega_t)\,]> \;/\; P(t) = 1 + E[\,r(t,T,\Omega_t)\,] \tag{2}$$

Thus, the dated information set comes down to an actual marker for both ex~ante and ex~post assessments. On the other hand, the different measures of both ex~ante and ex~post assessments of financial assets rates of return can be attempted because we take stock on different information sets

$$\Omega_t \quad , \quad \Omega_T$$

Contrasting both information is essential, because most of the time they are different. It is for information surprises to close the gap between them. Scholars and practitioners have lately become uneasy about this issue. With professor Elton's own words:

"*The use of average realized returns as a proxy for expected returns relies on a belief that information surprises tend to cancel out over the period of a study and realized returns are therefore an unbiased estimate of expected returns. However, I believe that there is ample evidence that this belief is misplaced"* (Elton, 1999).

Therefore, we need to set up foundations in the family of information sets so that different rates of return be properly defined. To accomplish such a task it seems advisable to resort to finite algebras of sets.

2. FINITE ALGEBRAS OF SETS

Given an arbitrary family, **A**, of subsets of the space **X**, it is very useful to endow it with structures in which operations between sets in **A** could be closed, meaning this that the outcome of the operation also belongs to the family **A**. Algebras of sets provide an example of such structures and come in handy for the purposes of this paper.

Definition 1 *Finite Algebras of Sets*

***A** is called a finite **algebra** of sets if it satisfies the following properties:*

$$\text{for every} \quad E \in \boldsymbol{A} \text{ and } \quad F \in \boldsymbol{A} \quad \Rightarrow \quad E \cup F \in \boldsymbol{A}$$

$$\text{for every} \quad E \in \boldsymbol{A} \quad \Rightarrow \quad E^C \in \boldsymbol{A}$$

In other words, a finite algebra of sets is an structure closed under finite unions and complements of sets. [Background on finite and infinite algebras in Halmos (1974) and Aliprantis (1999)]

Algebras are also closed for intersection of sets since, applying De Morgan's property, it follows that

$$\mathbf{E} \cap \mathbf{F} = [\,\mathbf{E}^C \cup \mathbf{F}^C\,]^C = [\,\mathbf{E}^C\,]^C \cap [\,\mathbf{F}^C\,]^C$$

When working with families of information sets it is very important to know whether there is or not a minimal algebra that contains the given family. The following lemma grants such a desirable feature.

Lemma 1 (on the existence of minimal algebras)

Let α be any class of sets in the space X. Then, there exists a minimal (smallest) algebra

$$A[\,\alpha\,]$$

which contains the family α.

Proof:

Certainly, the family α is contained in **P(X)**, the set consisting of all the subsets of **X,** which is the largest algebra available from **X**. Let **A[α]** the intersection of all algebras which contain the family α. For every E, F taken in **A[α]**, their union and complements belong to it because that is what happens in any algebra involved. Minimality follows outright because any algebra containing in **A[α]** it would also contain the family α and hence **A[α]**. *END*

It is usually said that the family α is covered by the algebra **A[α]** .

2.1. APPLICATION OF LEMMA 1 TO INFORMATION SETS

a) For any family α of information sets there exists a minimal algebra which contains α**.**

b) **X** could be translated as the set of all information available at the date " t ", and labeled in that case as

$$\mathbf{X} = \mathbf{X}_t$$

whereas the information set could be understood as the set of all available information (either private or public) within the reach of any economic agent $\mathbf{G}_{\lambda}$, where λ belongs to an index set **J** such that

$$\Omega_t\,(\mathbf{G}_{\lambda}) \subseteq \mathbf{X}_t \quad \text{, for every } \lambda \in \mathbf{J}$$

3. DIFFERENTIAL RATES OF RETURN

If we had access to information subsets of Ω_t, this would allow us to find out different sources which could explain the rate of change $\mathbf{r}(\,t\,,\,T\,,\,\Omega_t\,)$ as defined on the basic information set Ω_t and measured as in (1). What, for instance, if we knew that there is a subset $\Omega^1{}_t$ of Ω_t, namely,

$$\Omega^1{}_t \subseteq \Omega_t$$

which is so influential that the rate of change $\mathbf{s}^1\,(\mathbf{t}\,,\mathbf{T}\,,\Omega^1{}_t)$ could explain two thirds of the value of **r(.)** at least?

It is within this setting that it becomes advisable to split down the original rate of return into two components:

$$\mathbf{1 + r(t, T, \Omega_t) = [1 + s^1(t, T, \Omega^1_t)] . [1 + g^1(.)]} \qquad (3)$$

where

$$\mathbf{s^1(t, T, \Omega^1_t)}$$

stands for the rate of change conditional to the information set Ω^1_t. Furthermore, **$g^1(.)$** comes up by solving the equation and stands for whatever remains of **r(.)** after taking into account **$s^1(.)$**. So, this complementary rate **$g^1(.)$** fills the gap between both rates and it comes as conditional upon the information set

$$\Omega_t - \Omega^1_t$$

that reads " the set of elements in Ω_t which do not belong to Ω^1_t

It follows that

$$\mathbf{g^1(.) = g^1(t, T, \Omega_t - \Omega^1_t)}$$

Hence, we can now restate the former relationship between these three rates of change:

$$\mathbf{1 + r(t, T, \Omega_t) = [1 + s^1(t, T, \Omega^1_t)] . [1 + g^1(t, T, \Omega_t - \Omega^1_t)]} \qquad (4)$$

Although next definition keeps a very simple format lying on a general algebra of sets, it fulfills two roles. Firstly, it sums up what was developed above. Secondly, it paves the way for a more general format that will uncover, in section 4, the underlying structure of differential rates over residual information sets.

Definition 2 Simple Differential Rates over Residual Information Sets

*Let **A** be an algebra of sets in X_t. Given two information sets Ω^1_t and Ω_t, such that $\Omega^1_t \subseteq \Omega_t$ and rates of return $r(t, T, \Omega_t)$ and $s^1(t, T, \Omega^1_t)$,*

*it is said that **$g^1(.)$ is a simple differential rate to both r(.) and $s^1(.)$** if and only if*

$$g^1(.) = g^1(t, T, \Omega_t - \Omega^1_t)$$

and it fulfils:

$$1 + r(t, T, \Omega_t) = [1 + s^1(t, T, \Omega^1_t)] . [1 + g^1(t, T, \Omega_t - \Omega^1_t)]$$

*Furthermore, the set $\Omega_t - \Omega^1_t$ will be called a **residual information set for $g^1(.)$**.*

Residual information sets come into light only because we have an underlying algebra which provides closure for differences. In fact:

$$\Omega_t - \Omega^1{}_t = \{ (x \in \Omega_t) \text{ and } (x \notin \Omega^1{}_t) \} = \Omega_t \cap [\Omega^1{}_t]^C$$

What if we now tried to deal with **g^1 (.)** the same way we did with **r(.)**, by resorting to the simple differential rates definition? That is to say, what if knew that we can single out another rate **s^2 (.)** conditional upon the information set $\Omega^2{}_t$ that fulfills:

$$\Omega^2{}_t \subseteq \Omega_t \quad , \quad \Omega^2{}_t \cap \Omega^1{}_t = \varnothing$$

In that case we should ask for the remainder of **g^1 (.)**, and try to solve:

$$1 + g^1(t, T, \Omega_t - \Omega^1{}_t) = [1 + s^2(t, T, \Omega^2{}_t)] \cdot [1 + g^2(.)]$$

Recursively, we may eventually obtain a finite vector of rates of change

$$< s^1(.), s^2(.), \ldots\ldots, s^N(.) >$$

all of them stemming from the primary rate **r(.)**, and also a differential rate **g^N (.)** performing as a remainder, so as the following relationship must hold true by iteration:

$$1 + r(t, T, \Omega_t) = \{ \Pi_{1 \leq k \leq N} [1 + s^k(t, T, \Omega^k{}_t)] \} \cdot [1 + g^N(.)]$$

However, a problem is certain to arise with this expression, because we have still said nothing on the underlying residual information sets listed by next vector

$$< g^1(.), g^2(.), \ldots\ldots, g^N(.) >$$

To handle this difficulty, we need to enlarge upon Definition 2. However, we firstly have to agree on which sort of algebra sets will be used as from now.

4. LOOKING FOR CONTEXTUAL SETS AND CHOOSING THE RELEVANT ALGEBRA OF INFORMATION SETS

In practice, we choose a primary information set Ω_t in the space X_t and, at the same time, a ***family*** α ***of contextual sets to*** Ω_t ***in*** X_t, that is to say, a family of sets which amount or refer to contexts of information stemming from economic variables, transaction cost structures, institutional frameworks, that have ultimately connection or relevance to Ω_t.

It is the purpose of this section to show how a family of relevant sets may be spanned into a suitable minimal algebra. An example will shed light about contextual sets.

While dealing with financial assets rates of return, we should be intent on making the analysis inclusive of the transaction costs structure. At least, five contextual subsets of **X** seems to be useful (this approach has been developed in Apreda, 2000a and 2000b) :

- Intermediation costs : INT
- Microstructure costs : MICR
- Financial costs on transactions : FIN
- Information costs : INF
- Taxes : TAX

We define a family α of contextual sets to Ω_t :

$$\alpha = \{ E , F , G , H , J \}, \qquad \text{where}$$

$$E = \Omega_t \cap INT \,;\, F = \Omega_t \cap MICR \,;\, G = \Omega_t \cap FIN \,;\, H = \Omega_t \cap INF \,;\, J = \Omega_t \cap TAX$$

The minimal algebra which is spanned by these sets and Ω_t. will meet our purposes. Moreover, Lemma 1 grants its existence. We denote this algebra as $\mathbf{A}[\, \Omega_t , \alpha \,]$. That is to say:

$$\mathbf{A}[\, \Omega_t , \alpha \,] = \mathbf{A}[\, E, F, G, H, J, \Omega_t \,]$$

If we wanted to go beyond transaction costs, the class would be embedded into a larger one, with more factors of analysis; for instance, inflation, the rate of exchange, institutional constraints, even the perfect information set assumed by the Security Market Line in the CAPM world [this has been worked out in Apreda (2001a)]. Making for a definition, we get:

Definition 3 *The Relevant Algebra for Information Sets*

Given $\Omega_t \subseteq \mathbf{X}_t$ *, and a finite family of contextual sets* to Ω_t *in* $\mathbf{X}_t$ *,*

$$\alpha = \{ E_1 , E_2 , E_3 , \ldots.. , E_N \}$$

the algebra $\mathbf{A}[\, \Omega_t , \alpha \,]$ *spanned by the family*

$$\{ E_1 \cap \Omega_t , E_2 \cap \Omega_t , E_3 \cap \Omega_t , \ldots.. , E_N \cap \Omega_t \;,\; \Omega_t \}$$

*it will be called the **relevant algebra to the information set*** Ω_t *, subject to the contextual famiy* $\boldsymbol{\alpha}$*.*

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5. DIFFERENTIAL RATES OF RETURN OVER RESIDUAL INFORMATION SETS

Once we get a single differential rate, to what extent can we add other differential rates which could improve our knowledge of the primary rate, $\mathbf{r(t, T , \Omega_t)}$? Successive differential rates stemming from nested residual information sets, at each stage, claim for an iterative algorithm. That is to say, we have to deal with a sort of splitting-down decision problem, which is addressed by next lemma.

In order to avoid cumbersome steps in the development of the lemma below, a methodological short-cut has been adopted;

a) the recursive program is fully expanded on the Appendix at the end of this paper;
b) throughout the proof of the lemma, references to distinctive steps of the algorithm will be prompted any time they are needed.

Lemma 2 (on the generalized differential rates over residual information sets)

Let $A[\Omega_t , \alpha]$ be the relevant algebra to the information set Ω_t, indexed by a finite interval I of positive integers,

$$A[\Omega_t , \alpha] = \{ \Omega^p{}_t : p \in I \}$$

Then, the following statements hold true:

i) For every pair of rates of return $r(t ,T, \Omega_t)$ and $s^1(t ,T, \Omega^1{}_t)$ such that

$$\Omega^1{}_t \subseteq \Omega_t \in A[\Omega_t , \alpha]$$

there is one $g^1(\,.\,)$ which becomes a differential rate to both $r(\,.\,)$ and $s^1(\,.\,)$, and a well-defined matching residual information set for $g^1(\,.\,)$

ii) For every finite vector of rates of return,

$$< s^1(t ,T, \Omega^1{}_t), s^2(t ,T, \Omega^2{}_t), \ldots, s^k(t ,T, \Omega^k{}_t) >$$

$$\text{such that} \quad \Omega^j{}_t \in A[\Omega_t , \alpha] \qquad (j: 1,2, \ldots , k)$$

there is one $g^k(\,.\,)$ which becomes a differential rate concerning the preceding rates of return, and a well-defined matching residual information set for $g^k(\,.\,)$.

Proof:

i) From

$$1 + r(t, T, \Omega_t) = [\, 1 + s^1(t, T, \Omega^{p(1)}{}_t) \,] \,.\, [\, 1 + g^1(.) \,]$$

it holds that $\mathbf{g^1(\,.\,)}$ becomes a differential rate to both $r(\,.\,)$ and $s^1(\,.\,)$ by solving the equation outright. Besides, steps 1 to 4 in the recursive program (see Appendix) show how to construct a well-defined residual set matching this differential rate, which will be called $\Omega^{res(1)}{}_t$:

$$\Omega^{res(1)}{}_t = < \Omega_t - \Omega^{p(1)}{}_t > \cup < \cup \{ (\, \Omega_t \cap \Omega^p{}_t \,) : p \in I ; p \neq p(1) \} >$$

ii) We are going to proceed by induction on k.

If k = 1, then we take advantage of what we did in i) above, a procedure that is grounded on steps 1 to 5 from the recursive program (see Appendix).

Next, let us suppose that the statement ii) holds true for any k. We need to prove that it also holds true for (k + 1). If we take advantage of steps 6 to 8 from the recursive program, the (k + 1) equating differential rate should be

$$1 + r(t, T, \Omega_t) = \tag{5}$$

$$\{ \Pi_{1 \leq j \leq k+1} [1 + s^j(t, T, \Omega^{p(j)}{}_t)] \} . [1 + g^{k+1}(.)]$$

and the residual information set for $\mathbf{g^{k+1}(\,.\,)}$ would translate into this format:

$$\Omega^{res(k+1)}{}_t = \tag{6}$$

$$= < \Omega_t - \cup\{\Omega^{p(j)}{}_t : j = 1,2,\ldots,k+1\} > \cup < \cup \{ (\Omega_t \cap \Omega^p{}_t) : p \in I;\ p \neq p(1), p(2), \ldots, p(k+1) \} >$$

Firstly, by the inductive hypothesis, we notice that (5) implies

$$1 + r(t, T, \Omega_t) = \tag{7}$$

$$\{ \Pi_{1 \leq j \leq k} [1 + s^j(t, T, \Omega^{p(j)}{}_t)] \} . [1 + g^k(t, T, \Omega^{res(k)}{}_t)] . [1 + g^{k+1}(.)]$$

and then $\mathbf{g^{k+1}(\,.\,)}$ follows by solving (7).

Secondly, at this (k+1)-stage, we are going to pick another member of $\mathbf{A[\,\Omega_t\,,\alpha\,]}$

$$\Omega^{p(k+1)}{}_t \subseteq \Omega_t$$

which is relevant to account for the rate of return

$$s^{k+1}(t, T, \Omega^{p(k+1)}{}_t)$$

Up to this point, we would like to isolate (like in the recursive program) from $\Omega^{p(k+1)}{}_t$ any point which might be shared with another member of $\mathbf{A[\,\Omega_t\,,\alpha\,]}$ eventually. As we are at the (k+1)-stage, the previous selection process has brought about a string of residual information sets:

$$< \Omega^{p(1)}{}_t \ ; \ \Omega^{p(2)}{}_t \ ; \ \Omega^{p(3)}{}_t \ ; \ \ldots\ldots ; \ \Omega^{p(k)}{}_t >$$

Therefore, we should isolate from

$$\Omega^{p(1)}{}_t \cup \Omega^{p(2)}{}_t \cup \Omega^{p(3)}{}_t \cup \ldots\ldots \cup \Omega^{p(k+1)}{}_t$$

any point which could be shared with another member of $\mathbf{A}[\,\Omega_t\,,\alpha\,]$. This is accomplished by means of the set:

$$< \Omega^{p(1)}{}_t \cup \Omega^{p(2)}{}_t \cup \ldots \cup \Omega^{p(k+1)}{}_t > \; - \; \cup\{\Omega^{p}{}_t \;:\; p \neq p(1), p(2), \ldots, p(k+1)\}$$

Now we define the residual information set for the differential rate $\mathbf{g}^{k+1}(\,.\,)$ by means of

$$\Omega^{res(k+1)}{}_t =$$

$$= < \Omega_t - [\cup\{\Omega^{p(j)}{}_t : j = 1,2,\ldots,k+1\}> \cup < \cup\{(\Omega_t \cap \Omega^{p}{}_t): p \in I;\, p \neq p(1),p(2),\ldots,p(k+1)\}>$$

by following the same procedure as in steps 6 to 8 of the Appendix. But this is (6) and finishes the inductive proof. *END*

Remarkably, Lemma 2 enables us to clear up the meaning of differential rates over residual information sets, laying the grounds for the following definition.

Definition 4 *Differential Rates over Residual Information Sets*

Let $\mathbf{A}[\,\Omega_t\,,\alpha\,]$ *be the relevant algebra for the information set* Ω_t*, indexed by a finite interval* ***I*** *of positive integers,*

$$\mathbf{A}[\,\Omega_t\,,\alpha\,] = \{\,\Omega^{p}{}_t : p \in I\,\}$$

Given a finite vector of rates of return,

$$< s^1(t, T, \Omega^{p(1)}{}_t),\, s^2(t, T, \Omega^{p(2)}{}_t),\, \ldots,\, s^N(t, T, \Omega^{p(N)}{}_t) >$$

such that $\Omega^{p(j)}{}_t \subseteq \Omega_t$ *for j: 1, 2, ... , N*

it is said that ***$g^N(\,.\,)$ is a differential rate to $r(t, T, \Omega_t)$*** *within the framework of*

$$1 + r(t, T, \Omega_t) =$$

$$\{\,\Pi_{1\leq j\leq N}\,[\,1 + s^j(t, T, \Omega^{p(j)}{}_t)\,]\,\}\,.\,[\,1 + g^N(\,.\,)\,]$$

where its residual information set becomes

$$\Omega^{res(N)}{}_t =$$

$$= < \Omega_t - \cup\{\,\Omega^{p(j)}{}_t : j = 1,2,\ldots,N\,\}> \;\cup\; < \cup\{\,(\Omega_t \cap \Omega^{p}{}_t): p \in I;\, p \neq p(1),p(2),\ldots,p(N)\,\}>$$

It can be seen from the definition above that, for each value of N, a differential rate with its matching residual information set is produced. The definition, therefore, conveys a recursive feature.

6. TRANSACTIONAL ALGEBRAS A NON-STANDARD APPROACH TO ARBITRAGE

Either the standard trading arbitrage model or the financial arbitrage model, they both abstract from institutional and contractual arrangements, intermediaries and law enforcement, trading procedures and regulations, even transaction costs are left out. A non-standard approach to arbitrage has to redress this tight constraint, by making explicit the world where transactions take place eventually [background on this, with extensive analysis of arbitrage processes in Apreda (2003)].

We are going to introduce, therefore, the concept of a transactional algebra that intends to supply with a framework of analysis of financial trading, including arbitrage processes, embedding within it the institutional context, a convenient transaction costs function, and residual information sets.

Definition 5 Transactional Algebras

*We are going to call **Transactional Algebra** a complex structure whose distinctive features are:*

i. Existence of one or more markets where financial assets can be exchanged through the channels of public or private placements.

ii. An institutional framework and a market microstructure both set up by means of

trading and regulatory institutions;
intermediaries, investors, and regulators;
enforceable laws and rules of the game;
contractual arrangements about the property rights attached to each transaction.

iii. A total transaction costs function explicitly given which includes all computable enlarged transaction costs.

iv. Residual information sets and differential rates of return managed out of a relevant algebra of contextual sets

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6.1 ENLARGED TRANSACTION COSTS THE COSTS OF RUNNING TRANSACTIONAL ALGEBRAS

Transaction costs are usually neglected on the grounds of being small. Furthermore, as it is stated in some quarters, some transaction costs are becoming negligible as communication devices improve.

By all means, this belief is misplaced because transaction costs are the costs of running nothing less than a transactional algebra (on this regard, see Apreda 2000a, 2000b).

Firstly, what it is customarily meant by transaction costs points only at some particular types of trading costs, mainly linked with purchasing and selling securities. Although in some markets trading costs are being curbed, in other places they are not. The sensible question to elicit is about the structure of those trading costs, which is not so simple as it seems at first sight.

Secondly, enlarged transaction costs encompass a broad variety of items:

- intermediation (INT),
- microstructure (MICR),
- information (INF),
- taxes (TAX),
- and financial costs (FIN).

Although these five categories are neither exhaustive nor the only ones to work with, we believe that they allow for a sensible assessment of the transaction costs rate as

$$\mathbf{TC}\,(t_1\,;t_2\,;m_1\,;m_2\,;\ \Omega^{TC}{}_{t1})$$

which is a construct with the following features:

a) it comes by the side of every single transaction;
b) it amounts to a rate of change that may be expressed in percentage;
c) and it may be framed by means of the functional relationship of a multiplicative model:

$$< 1 + TC\,(t, T, \Omega^{TC}{}_t) > \; = \; < 1 + INT\,(t, T, \Omega^{INT}{}_t) > . < 1 + MICR(t, T, \Omega^{MICR}{}_t) > .$$

$$. < 1 + TAX\,(t, T, \Omega^{TAX}{}_t) > . < 1 + INF\,(t, T, \Omega^{INF}{}_t) > . < 1 + FIN\,(t, T, \Omega^{FIN}{}_t) >$$

with the restriction

$$\Omega^{k}{}_{t} \subseteq \Omega^{TC}{}_{t} \quad \text{for } k: \text{INT, MICR, INF, FIN, TAX}$$

where $\Omega^{k}{}_{t}$ stands for any distinctive subset of the underlying information set to the transaction costs rate.

Remark

Each component has its own functional structure which does not come up as linear, necessarily. In fact, non-linearity is customary and useful in standard research, which take advantage of piece-wise linear functions, or still better, the so called simple or step functions, so as to approximate more complex relationships. [For instance, Levy-Livingston (1995) on portfolio management, Day (1997) in nonlinear dynamics applied to economics, become helpful good sources]

We are interested here in applying this functional relationship to arbitrage processes, distinguishing long from short positions, as shown below. Bear in mind that when selling, costs lessen the cash flows to be finally collected; when purchasing, they add to incurring outflows.

$$\begin{cases} \textbf{short position:} \quad 1 + TC(t_1 ; t_2 ; m_1 ; m_2 ; s) = \\ [1 - int(s)] \,.\, [1 - micr(s)] \,.\, [1 - tax(s)] \,.\, [1 - inf(s)] \,.\, [1 - fin(s)] \\ \\ \textbf{long position:} \quad 1 + TC(t_1 ; t_2 ; m_1 ; m_2 ; l) = \\ [1 + int(s)] \,.\, [1 + micr(s)] \,.\, [1 + tax(s)] \,.\, [1 + inf(s)] \,.\, [1 + fin(s)] \end{cases} \tag{8}$$

Before concluding this section, we need to take a step further and embed each transaction cost rate pertaining the short and long position into a comprehensive differential rate that account for the costs of running transactional algebras.

Definition 6 *Differential Transaction Cost Rates*

In a transactional algebra setting, let us denote with

diff TC

the differential transaction costs rate that solves the equation

$$1 + diff\,TC = [1 + TC(t_1 ; t_2 ; m_1 ; m_2 ; s)] \,/\, [1 + TC(t_1 ; t_2 ; m_1 ; m_2 ; l)]$$

In other words, the differential transaction costs rate measures up the whole impact of the costs of running the transactional algebra involved with an arbitrage.

7. ARBITRAGE WITHIN TRANSACTIONAL ALGEBRAS

Standard **Financial Arbitrage** is a decision making process whose main features are:

i. the trade of a financial asset "g_1", at an expected moment "t_1", in a certain market "m_1", at the value $\mathbf{V(g_1; m_1; t_1)}$;

ii. the trade of a financial asset "g_2", at an expected moment "t_2", in a certain market "m_2", at the value $\mathbf{V(g_2; m_2; t_2)}$, with $\mathbf{t_1 \leq t_2}$;

iii. making a sure profit from round-off transactions, either the long-short or the short-long types, that is to say, the payoff functions $\Pi(\,.\,)$ are positive:

$$\begin{cases} \Pi(\text{long-short}) \;=\; V\,(g_2;\, m_2;\, t_2\,;\, s\,) \;/\; V(g_1;\, m_1;\, t_1\,;\, l\,) \;>\; 1 \\ \Pi(\text{short-long}) \;=\; V\,(g_1;\, m_1;\, t_1\,;\, s\,) \;/\; V(g_2;\, m_2;\, t_2\,;\, l\,) \;>\; 1 \end{cases}$$

iv. no investment is required for rounding off both transactions;

v. risks of rounding off both transactions are null.

Remark

Background on Arbitrage in Dybgiv and Ross(1992) while trading and financial arbitrage in transactional algebras is taken up in Apreda (2003).

Within a transactional algebra, rates of return must be broken down into the cost components on the one hand, and a return netted from them, on the other hand. To deal with this issue two lemmas follow.

Lemma 3 (on the existence of an arbitrage return net of transaction costs)

In a transactional algebra, and for every arbitrage opportunity:

i) there is an arbitrage return net of transaction costs.

ii) furthermore, there are diferential rates to translate each type of transaction costs arising from the rounding-off trades;

Proof:

i) The money to collect when selling the asset (from an all-in-cost basis) would amount to

$$V(\,s\,)\,.\,[\,1 - int(s)\,]\,.\,[\,1 - micr(s)\,]\,.\,[\,1 - tax(s)\,]\,.\,[\,1 - inf(s)\,]\,.\,[\,1 - fin(s)\,] \;= \tag{9}$$

$$=\; V(\,s\,)\,.\;[\,1 \;+\; TC\,(\,t_1\,;\,t_2\,;\,m_1\,;\,m_2\,;\,s\,)\,]$$

and when purchasing the asset (from an all-in-cost basis):

$$V(\,l\,)\,.\,[\,1 \;+\; int(l)\,]\,.\,[\,1 \;+\; micr(l)\,]\,.\,[\,1 \;+\; tax(l)\,]\,.\,[\,1 +\; inf(l)\,]\,.\,[\,1 \;+ fin(l)\,] \;= \tag{10}$$

$$=\; V(\,l\,)\,.\,[\,1 \;+\; TC\,(\,t_1\,;\,t_2\,;\,m_1\,;\,m_2\,;\,l\,)\,]$$

The return that this arbitrage yields (either trading or financial, and in nominal terms) follows from

$$V(s) \;/\; V(l) \;=\; 1 \;+\; r(\text{arbitrage})$$

When taking transaction costs arising from the rounding-off trades, as in (9) and (10), we can put forth (8) into the shape of

$$\mathbf{V(s) \,.\, [\,1 + TC\,(\,t_1\,;\,t_2\,;\,m_1\,;\,m_2\,;\,s\,)\,] \;/\; V(\,l\,)\,.\,[\,1 + TC\,(\,t_1\,;\,t_2\,;\,m_1\,;\,m_2\,;\,l\,)\,] =} \tag{11}$$

$$\mathbf{= \; 1 \; + \; r_{net} \; (arbitrage)}$$

and, solving for r_{net} (arbitrage) we have a measure of the return of the arbitrage that embodies transaction costs.

By resorting to definition 6, the differential transaction costs rate follows from

$$\mathbf{1 + diff\,TC \;=\; [\,1 + TC\,(\,t_1\,;\,t_2\,;\,m_1\,;\,m_2\,;\,s\,)\,] \;/\; [\,1 + TC\,(\,t_1\,;\,t_2\,;\,m_1\,;\,m_2\,;\,l\,)\,]} \tag{12}$$

and the relationship (11) can now be rewritten in a much more compact format:

$$\mathbf{[\,V(s) \;/\; V(\,l\,)\,]\,.\,(\,1 \;+\; diff\,TC\,) \quad = \quad 1 \;+\; r_{net} \; (arbitrage)}$$

Last of all, we get

$$\mathbf{[\,1 + r(arbitrage)\,] \;\;.\;\; (\,1 \;+\; diff\,TC\,) \quad = \quad 1 \;+\; r_{net} \; (arbitrage)}$$

ii) Let us substitute now the labels t_k (k: 1, 2, 3, 4, 5) for the enlarged transaction costs labels trad, micr, tax, inf, and fin, respectively. Then,

$$\mathbf{1 + g_k \;=\; [\,1 \;-\; t_k\,(s)\,] \;/\; [\,1 \;+\; t_k\;(l)\,]}$$

where $\boldsymbol{g_k}$ performs as a differential rate drawn out of $\boldsymbol{t_k\,(s)}$ and $\boldsymbol{t_k\;(l)}$.

On the other hand, we can replace in (12) to get the equivalence:

$$\mathbf{1 \;+ diff\,TC \quad = \quad [\;\prod\;(\,1 + g_k\,)\;] \;\; =}$$

$$\mathbf{= [\,1 \;+\; TC\,(\,t_1\,;\,t_2\,;\,m_1\,;\,m_2\,;\,s\,)\,] \;/\; [\,1 \;+\; TC\,(\,t_1\,;\,t_2\,;\,m_1\,;\,m_2\,;\,l\,)\,]} \qquad \textit{END}$$

Although in the standard financial arbitrage model arbitrage opportunities can be grabbed regardless of transactional features, in a transactional algebra structure this cannot be granted. In fact, arbitrage will only be feasible whenever the arbitrage gap overreaches the constraints of the transactional algebra, as the following lemma makes clear.

Lemma 4 (on conditions for an arbitrage to take place in a transactional algebra)

In a transactional algebra setting, to the fulfillment of the standard financial arbitrage conditions it must also be added the following boundary conditions

$$\begin{cases} \boldsymbol{1 + r_{net}\,(arbitrage;\ long\text{-}short) \;=\; [\,1 + r(arbitrage);\ long\text{-}short\,]\,.\,[\,1 + diff\ TC\,]) \quad > 1} \\ \boldsymbol{1 + r_{net}\,(arbitrage;\ short\text{-}long) \;=\; [\,1 + r(arbitrage);\ short\text{-}long\,]\,.\,[\,1 + diff\ TC\,] \quad > 1} \end{cases}$$

Proof:

Whenever an investor takes advantage of arbitrage opportunities, he tries to lock in a sure profit that follows from

$$\mathbf{1 + r(arbitrage;\ long\text{-}short) \quad = \quad V\,(g_2;\ m_2;\ t_2\ ;\ s\,) \;\;/\;\; V(g_1;\ m_1;\ t_1\ ;\ I\,)}$$

$$\mathbf{1 + r(arbitrage;\ short\text{-}long) \quad = \quad V\,(g_1;\ m_1;\ t_1\ ;\ s\,) \;\;/\;\; V(g_1;\ m_2;\ t_2\ ;\ I\,)}$$

Let us analyze each relationship at a turn.

a) *<u>the long-short type of arbitrage</u>*

If we included transaction costs, according with Lemma 3, we would get a net arbitrage return:

$$\mathbf{1 \; + \; r_{\,net}\;(arbitrage) \quad = \quad [\,1 + r(arbitrage)\,] \;\; . \; [\,1 \; + \; diff\ TC\,]}$$

For the arbitrage to become successful, it must hold:

$$\mathbf{[\,1 \; + r\,(arbitrage;\ long\text{-}short)\,] \;=\; [\,V\,(g_2;\ m_2;\ t_2\ ;\ s\,) \;/\; V(g_1;\ m_1;\ t_1\ ;\ I\,)\,] \;\; > 1}$$

but this is not a sufficient feature, because differential transaction costs could lead to

$$\mathbf{1 + r_{net}\,(arbitrage;\ long\text{-}short) \;=\; [\,1 + r(arbitrage);\ long\text{-}short\,]\,.\,[\,1 + diff\ TC\,]) \quad < 1}$$

thus yielding a negative return on the net rate of return for the arbitrage.

b) *<u>the short-long type of arbitrage</u>*

By the same procedure as in a) we would get that is not enough the fulfillment of

$$\mathbf{[\,V\,(g_1;\ m_1;\ t_1\ ;\ s\,) \;/\; V(g_2;\ m_2;\ t_2\ ;\ I\,)\,] \;\; > 1}$$

because the transaction costs structure could bring about the following outcome:

$$\mathbf{1 + r_{net}\,(arbitrage;\ short\text{-}long) \;=\; [\,1 + r(arbitrage);\ short\text{-}long\,]\,.\,[\,1 + diff\ TC\,] \quad < 1}$$

giving forth a negative return on the net rate of return for the arbitrage. *END*

Profitting from Lemma 2, we can frame a definition of what is meant by financial arbitrage within a transactional algebra.

Definition 7 Financial Arbitrage within a Transactional Algebra

*In a transactional algebra setting, **Financial Arbitrage Algebra** is a decision-making process whose main features are:*

i. the trade of a financial asset "g_1", at an expected moment "t_1", in a certain market "m_1", at the value $\boldsymbol{V(g_1; m_1; t_1)}$ *;*

ii. the trade of a financial asset "g_2", at an expected moment "t_2", in a certain market "m_2", at the value $\boldsymbol{V(g_2; m_2; t_2)}$, with $\boldsymbol{t_1 \leq t_2}$ *;*

iii. making a sure profit from round-off transactions, either the long-short or the short-long types, that is to say, the payoff functions $\Pi(\,.\,)$ are positive:

$$\begin{cases} \Pi(\text{long-short}) = 1 + r(\text{arbitrage; long-short}) = V(g_2; m_2; t_2\,; s\,) \,/\, V(g_1; m_1; t_1\,; l\,) > 1 \\ \Pi(\text{short-long}) = 1 + r(\text{arbitrage; short-long}) = V(g_1; m_1; t_1\,; s\,) \,/\, V(g_1; m_2; t_2\,; l\,) > 1 \end{cases}$$

iv. no investment is required for rounding off both transactions;

v. risks of rounding off both transactions are null.

vi. it meets the following boundary conditions

$$\begin{cases} 1 + r_{net}(\text{arbitrage; long-short}) = [\,1 + r(\text{arbitrage}); \text{long-short}\,] \,.\, [\,1 + \text{diff TC}\,]) > 1 \\ 1 + r_{net}(\text{arbitrage; short-long}) = [\,1 + r(\text{arbitrage}); \text{short-long}\,] \,.\, [\,1 + \text{diff TC}\,] > 1 \end{cases}$$

7.1. PRACTICAL CONSEQUENCES OF LEMMA 1 AND LEMMA 2

We are going to point out to a pair of consequences that are worthy of further comment.

a) Lemma 3 brings about a strong outcome:

$$\mathbf{1 + r_{net}\,(arbitrage) = [\,1 + r(arbitrage)\,] \,.\, [\,1 + diff\ TC\,]}$$

that is to say, in order to make a profitable arbitrage within a transactional algebra we should check out whether the nominal gross gap promised by the arbitrage opportunity is wide enough to cover the differential transaction costs rate. And Lemma 4 shows that decision-making follows only when this check is carried out eventually.

On the other hand, it is expected most of the time that

$$\mathbf{1 + diff\ TC < 1}$$

because the final blend of long and short positions (regarded from the transaction costs involved with them), it makes the differential rate negative, meaning that the final action of the transactional algebra amounts to lessening the gross arbitrage gap. But this boundary condition does not prevent some particular situations in which the differential rate become positive (in this case the costs of the long position are negligible or becomes negative); if such were the environment, the arbitrage rate would be reinforced, by all means.

b) From Lemma 3 we know that

$$\mathbf{1 + r_{net}\ (arbitrage) = [\,1 + r(arbitrage)\,]\ .\ [\,1 + diff\ TC\,]}$$

that allows for an equivalent translation by applying the concept of reverse differential rate to **diff TC**,

$$\mathbf{[\,1 + diff\ TC\,]\ .\ [\,1 + rev\ diff\ TC\,] = 1}$$

and solving for the reverse of **diff TC**

$$\mathbf{[\,1 + rev\ diff\ TC\,] = [\,1 + diff\ TC\,]^{-1}}$$

we can have a symmetric copy of the main outcome conveyed by Lemma 3:

$$\mathbf{[\,1 + r(arbitrage)\,] = [\,1 + r_{net}\ (arbitrage)\,]\,.\,[\,1 + rev\ diff\ TC\,]}$$

This outcome can be used as follows: in order to have a profitable arbitrage, the joint action of the differential transaction costs and the minimal gap that makes the arbitrage worthy of being carried out, must be at least as big as the expected gross arbitrage gap.

CONCLUSIONS

The main contributions of the paper can be summarized this way:

- It provides a discrete modelling framework, lying on finite algebras of sets, to deal with information sets. In this way, primary information sets linked with a rate of return can be broken down into strings of residual information sets. For instance, this is certainly the case with transaction costs structures.

- It defines contextual sets and the relevant algebra for information sets by which a recursive algorithm determines the structure of both generalized differential rates and residual sets.

- The concept of transactional algebra is introduced for the first time in the literature, with an outright application to financial arbitrage, establishing both existence and boundary conditions for the financial arbitrage to take place within a transactional algebra.

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APPENDIX: A RECURSIVE ALGORITHM TO SET UP DIFFERENTIAL RATES OVER RESIDUAL INFORMATION SETS

Step 1: Firstly, we index the algebra components at valuation date " t ".

$$\mathbf{A}[\,\Omega_t\,,\alpha\,] \;=\; \{\;\Omega^{p}{}_{t} : \mathbf{p} \in \mathbf{I}\;\}$$

where **I** is a finite interval of positive integers.

Step 2: Next, we pick up a member $\Omega^{p(1)}{}_{t} \subseteq \Omega_t$ in the algebra $\mathbf{A}[\,\Omega_t\,,\alpha\,]$, which is relevant to account for the rate of return

$$\mathbf{s}^{1}(\,\mathbf{t}\,,\mathbf{T}\,,\;\Omega^{p(1)}{}_{t})$$

Now, we want to exclude from $\Omega^{p(1)}{}_t$ any point which could be shared with another member of the algebra **A[** Ω_t **,** α **].** This lead to the set

$$\Omega^{p(1)}{}_t - \cup\{ \Omega^p{}_t : p \in I ; p \neq p(1) \} \tag{A1}$$

Step 3: It is a well known property of any arbitrary collection of sets defined in certain universe that

$$\begin{aligned} \mathbf{B} - [\, \mathbf{A}_1 \cup \mathbf{A}_2 \cup \mathbf{A}_3 \cup \ldots\ldots \cup \mathbf{A}_N \,] &= \\ = (\mathbf{B} - \mathbf{A}_1) \cap (\mathbf{B} - \mathbf{A}_2) \cap (\mathbf{B} - \mathbf{A}_3) \cap \ldots\ldots\ldots \cap (\mathbf{B} - \mathbf{A}_N) & \\ = \cap\, \{ (\mathbf{B} - \mathbf{A}_k) : k = 1, 2, 3, \ldots., N \} & \end{aligned} \tag{A2}$$

By linking (A2) to (A1), we get:

$$\begin{aligned} \Omega^{p(1)}{}_t - \cup\{ \Omega^p{}_t : p \in I ; p \neq p(1) \} &= \\ = \cap\{ (\Omega^{p(1)}{}_t - \Omega^p{}_t) : p \in I ; p \neq p(1) \} & \end{aligned} \tag{A3}$$

Step 4 : In the equation

$$1 + r(t, T, \Omega_t) = [\, 1 + s^1(t, T, \Omega^{p(1)}{}_t) \,] \,.\, [\, 1 + g^1(\,.\,) \,]$$

and taking advantage of (A3), we will choose as the underlying residual information set for $\mathbf{g^1(\,.\,)}$

$$\Omega^{res(1)}{}_t = \Omega_t - \cap\{ (\Omega^{p(1)}{}_t - \Omega^p{}_t) : p \in I ; p \neq p(1) \} \tag{A4}$$

This set brings back all the elements in $\Omega^{p(1)}{}_t$ which could be of interest for any other information set relevant to following differential rates. That is to say, $\mathbf{g^1(\,.\,)}$ is defined on all the points of Ω_t including those points in the sets overlapping with $\Omega^{p(1)}{}_t$.

Step 5 : We labour (A4) by means of properties that hold for arbitrary collections of sets, namely:

$$\begin{aligned} &\Omega_t - \cap\{ (\Omega^{p(1)}{}_t - \Omega^p{}_t) : p \in I ; p \neq p(1) \} = \\ &= \Omega_t \cap < \cap\{ (\Omega^{p(1)}{}_t - \Omega^p{}_t) : p \in I ; p \neq p(1) \} >^C \\ &= \Omega_t \cap < \cup\{ (\Omega^{p(1)}{}_t - \Omega^p{}_t)^C : p \in I ; p \neq p(1) \} > \\ &= \Omega_t \cap < \cup\{ (\Omega^{p(1)}{}_t \cap (\Omega^p{}_t)^C)^C : p \in I ; p \neq p(1) \} > \\ &= \Omega_t \cap < \cup\{ ((\Omega^{p(1)}{}_t)^C \cup \Omega^p{}_t) : p \in I ; p \neq p(1) \} > \\ &= < \Omega_t \cap (\Omega^{p(1)}{}_t)^C > \cup < \cup\{ (\Omega_t \cap \Omega^p{}_t) : p \in I ; p \neq p(1) \} > \\ &= < \Omega_t - \Omega^{p(1)}{}_t > \cup < \cup\{ (\Omega_t \cap \Omega^p{}_t) : p \in I ; p \neq p(1) \} > \end{aligned}$$

Summing up, the residual information set for $\mathbf{g^1(\,.\,)}$ will be:

$$\Omega^{res(1)}{}_t = \Omega_t - \cap\{ (\Omega^{p(1)}{}_t - \Omega^{p}{}_t) : p \in I ; p \neq p(1) \} =$$
$$= < \Omega_t - \Omega^{p(1)}{}_t > \cup < \cup\{ (\Omega_t \cap \Omega^{p}{}_t) : p \in I ; p \neq p(1) \} > \qquad (A5)$$

This relationship is worthy of being remarked:

(a) The first part of it, namely

$$< \Omega_t - \Omega^{p(1)}{}_t >$$

is the residual set mirrored from a narrow perspective, because only allows for non-overlapping sets.

(b) The second part of it, namely

$$< \cup\{ (\Omega_t \cap \Omega^{p}{}_t) : p \in I ; p \neq p(1) \} >$$

conveys the sharing of information with all overlapping sets, as stand-by remainders to be used by other differential rates.

Step 6 : At this stage, we are going to pick another member of $\mathbf{A[\,\Omega_t , \alpha\,]}$,

$$\Omega^{p(2)}{}_t \subseteq \Omega_t,$$

which is relevant to account for the rate of return

$$\mathbf{s^2 (t , T , \Omega^{p(2)}{}_t)}$$

Up to this point, we would like to isolate from $\Omega^{p(2)}{}_t$ any point which could be shared with another member of $\mathbf{A[\,\Omega_t , \alpha\,]}$ eventually. As we are in the second rate of return, another $\Omega^{p(1)}{}_t$ has been previously taken into account. So, we want to isolate from

$$\Omega^{p(1)}{}_t \cup \Omega^{p(2)}{}_t$$

any point which could be shared with another member of $\mathbf{A[\,\Omega_t , \alpha\,]}$. This leads to the set

$$(\Omega^{p(1)}{}_t \cup \Omega^{p(2)}{}_t) - \cup\{ \Omega^{p}{}_t : p \in I ; p \neq p(1) , p(2) \}$$

By using (A2) this set can also be translated as

$$(\Omega^{p(1)}{}_t \cup \Omega^{p(2)}{}_t) - \cup\{ \Omega^{p}{}_t : p \in I ; p \neq p(1) , p(2) \} =$$
$$= \cap\{ [(\Omega^{p(1)}{}_t \cup \Omega^{p(2)}{}_t) - \Omega^{p}{}_t] : p \in I ; p \neq p(1) , p(2) \} \qquad (A7)$$

Step 7 : Next, we have to solve for the new differential rate, as we did in step 4:

$$\mathbf{1 + g^1 (.) = [1 + s^2 (t , T , \Omega^{p(2)}{}_t)] . [1 + g^2 (.)]}$$

and we proceed to choose the residual information set of $\mathbf{g^2 (.)}$

$$\Omega_t - \cap\{ [(\Omega^{p(1)}{}_t \cup \Omega^{p(2)}{}_t) - \Omega^{p}{}_t] : p \in I ; p \neq p(1) , p(2) \} \qquad (A8)$$

This set brings back all the elements in $(\Omega^{p(1)}{}_t \cup \Omega^{p(2)}{}_t)$ which could be of interest for any other information set relevant to the following differential rates. That is to say, $\mathbf{g^2(\ .\ )}$ is defined on points of Ω_t including those points in the sets overlapping with

$$(\Omega^{p(1)}{}_t \cup \Omega^{p(2)}{}_t).$$

By working out (A8), through the same procedure followed in step 5, we get access to the residual information set of $\mathbf{g^2(\ .\ )}$:

$$\Omega^{res(2)}{}_t = \Omega_t - \cap\{ [(\Omega^{p(1)}{}_t \cup \Omega^{p(2)}{}_t) - \Omega^p{}_t] : p \in I ; p \neq p(1), p(2) \} = \tag{A9}$$

$$= < \Omega_t - (\Omega^{p(1)}{}_t \cup \Omega^{p(2)}{}_t) > \cup < \cup\{ (\Omega_t \cap \Omega^p{}_t) : p \in I ; p \neq p(1), p(2) \} >$$

It should not come as a surprise that the first part in the last expresion, namely:

$$< \Omega_t - (\Omega^{p(1)}{}_t \cup \Omega^{p(2)}{}_t) >$$

is similar to (a) in step 5, that is to say the same sort of residual set used in when dealing simple differential rates. On the other hand, the second part in the last expression amounts to

$$< \cup\{ (\Omega_t \cap \Omega^p{}_t) : p \in I ; p \neq p(1), p(2) \} >$$

it conveys sharing of information with overlapping sets, as stand-by remainders to be used by other differential rates.

Step 8 : As from now, the iteration of order k follows the pattern

i) differential rate $\mathbf{g^k(\ .\ )}$

$$1 + r(t, T, \Omega_t) = \tag{A10}$$

$$\{ \Pi_{1 \leq j \leq k} [1 + s^j(t, T, \Omega^{p(j)}{}_t)] \} . [1 + g(\ .\ )]$$

ii) residual information set matching the rate $\mathbf{g^k(\ .\ )}$

$$\Omega^{res(k)}{}_t = \Omega_t - \cap\{ [\cup\{ \Omega^{p(j)}{}_t : j = 1, 2, \ldots, k\} - \Omega^p{}_t] : p \in I ; p \neq p(1), p(2), \ldots p(k) \} = \tag{A11}$$

$$= < \Omega_t - \cup\{ \Omega^{p(j)}{}_t : j = 1,2, \ldots ,k\} > \cup < \cup\{ (\Omega_t \cap \Omega^p{}_t) : p \in I ; p \neq p(1), p(2), \ldots ,p(k) \} >$$