Regime-Switching Stochastic Volatility and Short-Term Interest Rates

Madhu Kalimipalli §
School of Business and Economics
Wilfrid Laurier University
Waterloo, Ontario N2L 3C5
Tel: 519-884-0710 Ext.: 2187
Mkalimip@wlu.ca

and

Raul Susmel
C.T. Bauer College of Business
University of Houston, Houston,Tx-77204-6282, USA
rsusmel@uh.edu

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Abstract

In this paper, we introduce regime-switching in a two-factor stochastic volatility model to explain the behavior of short-term interest rates. The regime-switching stochastic volatility (RSV) process for interest rates is able to capture all possible exogenous shocks that could be either discrete, as occurring from possible changes in the underlying regime, or continuous in the form of `market-news' events. We estimate the model using a Gibbs Sampling based Markov Chain Monte Carlo algorithm that is robust to complex non-linearities in the likelihood function. We compare the performance of our RSV model with the performance of other GARCH and stochastic volatility two-factor models. We evaluate all models with several in-sample and out-of-sample measures. Overall, our results show a superior performance of the RSV two-factor model.

Key Words: Short-term interest rates, stochastic volatility, regime switching, MCMC methods.

JEL Classification: G12

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In this paper, we introduce regime-switching in a two-factor stochastic volatility model to explain the behavior of short-term interest rates. The regime-switching stochastic volatility (RSV) process for interest rates is able to capture all possible exogenous shocks that could be either discrete, as occurring from possible changes in the underlying regime, or continuous in the form of `market-news' events. We estimate the model using a Gibbs Sampling based Markov Chain Monte Carlo algorithm that is robust to complex non-linearities in the likelihood function. We compare the performance of our RSV model with the performance of other GARCH and stochastic volatility two-factor models. We evaluate all models with several in-sample and out-of-sample measures. Overall, our results show a superior performance of the RSV two-factor model.

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I. Introduction

The volatility of short-term interest rates plays a crucial role in many popular two-factor models of the term structure. The level and the volatility of the short rate are commonly used as state variables in two-factor models. For example, Longstaff and Schwartz (1992) derive a two-factor general equilibrium model, with the short rate's level and the short rate's conditional volatility as factors. They show that a two-factor model improves upon a single factor model, which only uses the level of the short rate. They find that the conditional volatility factor provides additional information about the term structure that may be useful in pricing interest rate options and hedging interest rate risk. Similarly, Brenner et al. (1996) include a level effect and a GARCH effect into their interest rate model. They find that models with both level and GARCH effects outperform those that exclude one of them. Note that a GARCH model displays a single continuous information shock; while in a stochastic volatility (SV) model there are two continuous information shocks. Following this more general formulation for the conditional variance, Anderson and Lund (1997) and Ball and Torous (1999) include a level factor and a stochastic volatility factor into the interest rate mean specification. They find a two-factor model with stochastic volatility performs better than the more traditional two-factor model with GARCH volatility. The information shocks in both GARCH and SV models are continuous. Ball and Torous (1995) build a two-factor model, but introducing discrete shocks from an underlying state variable that follows a two-state Markov process. In Ball and Torous (1995), the conditional volatility displays Hamilton's (1989) regime-switching.

Introducing regime-switching in the volatility process of the short-term interest rate is consistent with previous studies that document a strong evidence for regime-switching in short-term interest rates (see Hamilton (1988), Driffill (1992) and Gray (1996)). Regime-switching in the volatility process of the short rate has important implications for the dynamics of the yield curve and immunization strategies. As pointed out by Litterman, Scheinkman and Weiss (1991), the volatility of the short rate (for example, three-month T-Bill rate) affects the curvature of yield curve[1]. In

[1] See also Brown and Schaefer (1995).

particular, the volatility of the short rate has two counteracting effects on the yield curve. First, higher volatility of the short rate induces higher expected rates for the longer maturities (premium effect). Second, higher volatility of the short-term interest rate increases the convexity of the discount factor function and, therefore, reduces the yields for longer maturities (convexity effect). The premium effect dominates at the short end of the yield curve, while the convexity effect dominates at the long end making the yield curve more humped. When regime switching is not considered, volatility shocks tend to be very persistent and, therefore, the convexity effect and the hump in the yield curve could be more pronounced than they ought to be. Gray (1996) notes that there is evidence for 1) mean reverting high-volatility state with low volatility persistence, and 2) non-mean reverting low-volatility state with high volatility persistence in one-month U.S. T-Bill yields. This implies that the shape of the yield curve depends upon the dynamics of the short rate, its volatility and the latent volatility state.

Regime-switching in the volatility process also has important implications for hedging. A trader should account for both continuous and discrete shocks to volatility in computing dynamic hedge ratios. While continuous shocks refer to market-news events, discrete shocks could refer to the high or low volatility states of the market, high or low liquidity in the market or high or low sentiment in the market.

In this paper, we follow Ball and Torous (1995) and Anderson and Lund (1997). We introduce regime-switching in a two-factor model, where volatility follows a SV process. We model the volatility of short-term interest rates as a stochastic process whose mean is subject to shifts in regime. That is, our switching stochastic volatility for interest rates captures all possible exogenous shocks that could be either discrete, as occurring from possible changes in underlying regime, or continuous, in the form of "market news" events. We estimate our two-factor regime-switching stochastic volatility model for short-term interest rates using a Gibbs Sampling based Markov Chain Monte Carlo algorithm. We conduct an extensive in-sample and out-of-sample evaluation of our two-factor model against other two-factor models. In-sample, our model performs substantially better than the GARCH based two-factor models and the single-state stochastic volatility two-factor models. Out-of-sample, the regime-switching stochastic volatility model tends to outperform the other models. The out-of-sample forecasts from the regime-switching stochastic volatility model, however, are not that different from the single-state stochastic volatility model.

The rest of the paper is structured as follows. Section II introduces our regime-switching stochastic volatility (RSV) two-factor model. Section III examines the data set used in this paper. Section IV discusses the results from estimation. Section V presents the in-sample and out-of-sample comparative performance of the RSV model. Section VI summarizes and presents our conclusions.

II. Two-Factor Models and Regime Switching

A common empirical finding in two-factor models is the high persistence in the conditional variance. For example, Brenner et al. (1996) estimate the persistence parameter in the conditional variance equation to be 0.82 using weekly three-month U.S. T-Bill data. Ball and Torous (1999) report persistence parameter to be 0.928 using monthly one-month U.S. T-Bill data, while Anderson and Lund (1997) report volatility persistence to be 0.98 for weekly three-month U.S. T-Bill data.

High persistence in the conditional variance implies that shocks to the conditional variance do not die out quickly -i.e., current information has a significant effect on the conditional variance for future horizons. Lamoreux and Lastrapes (1990) show that high persistence could be related to possible structural changes that have occurred during the sample period in the variance process. They find that a single-regime GARCH specification leads to spurious high persistence under the presence of structural breaks. By allowing for possible regime switching in the data, high persistence observed in the single regime models seems no longer valid. Similar results have been documented by Hamilton and Susmel (1994), Cai (1994) and So, Lam and Li (1998).

Hamilton (1988) and Driffill (1992), among others, find strong evidence for regime-switching in the U.S. short-term interest rates. Various macro-economic events were responsible for regime switching in the U.S. interest rates. These events include the OPEC oil crisis, the Federal Reserve experiment of 1979-82, the October 1987 crash and wars involving U.S. and rest of the world. When short rates could switch randomly between different regimes –i.e., where each regime is associated with its own mean and variance-, we may find high persistence in the data when we average data from different regimes. It is the possibility of a shift in the underlying regime that we explicitly incorporate in our short-rate process. Next, we introduce a two-factor model that nests level and stochastic volatility effects.

Consider the short-term interest rate process described below:

$$
\begin{aligned}
& r_t - r_{t-1} = \alpha_0 + \alpha_1 r_{t-1} + \sqrt{h_t r_{t-1}^{2\alpha}}\,\varepsilon_t, \qquad \alpha > 0 \\
& \left(\ln(h_t) - \mu\right) = \phi_1\left(\ln(h_{t-1}) - \mu\right) + \sqrt{\sigma_\eta^2}\;\eta_{t-1} \\
& \mu = x'_t\,\beta
\end{aligned}
\qquad (1)
$$

In model (1), r_t is the short rate and h_t is the conditional variance of the short rate, α captures the levels effect in the model, μ is the stationary mean of the log conditional, ϕ_1 measures the degree of persistence of conditional variance, and ε_t and η_t represent shocks to the mean and to the volatility, respectively. Both shocks are white noise errors, which are assumed to be distributed independently of each other. We call model (1) the Single-state Stochastic Volatility (SSV) model. This model is used in Ball and Torous (1999) and in Anderson and Lund (1997). The estimation of SSV models involves the estimation of mean parameters $\{\alpha_0, \alpha_1\}$ and variance parameters $\{\alpha, \beta, \sigma_\eta, \phi_1\}$. Note that model (1) with a GARCH specification instead of a SV specification for the conditional volatility becomes the Brenner et al. (1996) model.

Now, let μ be a function of the latent state s_t, which follows a k-state ergodic discrete first-order Markov process as in Hamilton (1988). That is, at a given point in time, the mean of the log volatility belongs to one of the k states. A k-state stationary transition probability matrix governs the dynamics of the transition from one state to the next state. This implies that the latent volatility, h_t, is driven by a continuous shock, η_t, as in (1) above and also by a discrete shock s_t that takes on discrete integer values {1, 2, ..., k}. We can also think of our latent volatility as a mixture of k densities, where each density corresponds to a single state. The latent volatility at a given time comes from a single density, which is decided by an underlying k-state Markov process. That is, our Regime-switching Stochastic Volatility (RSV) model is given by:

$$
\begin{aligned}
& r_t - r_{t-1} = \alpha_0 + \alpha_1 r_{t-1} + \sqrt{h_t r_{t-1}^{2\alpha}}\,\varepsilon_t, \qquad \alpha = 0.5 \\
& \left(\ln(h_t) - \mu_{s_t}\right) = \phi_1\left(\ln(h_{t-1}) - \mu_{s_{t-1}}\right) + \sqrt{\sigma_\eta^2}\;\eta_{t-1} \\
& \mu_{s_t} = \beta + \gamma s_t \qquad \gamma > 0 \qquad s_t = \{1,2,\ldots k\}
\end{aligned}
\qquad (2)
$$

where $\mu_{s(t)}$ refers to the state dependent mean of conditional volatility. The parameter γ measures the sensitivity of the mean with respect to the underlying state and is constrained to be positive. The underlying state s_t can assume k possible states, i.e. one of {1,2, ...,k} where higher values of s_t lead to higher intercept terms in the log variance equation. As an identification condition, we require each regime to correspond to at least one time point. In addition, and mainly for convenience, we set the level parameter α=0.5, which has been used in many previous papers.[2] This assumption also avoids potential non-stationary problems associated with $\alpha > 1$, as shown in Gray (1996) and Bliss and Smith (1998). The estimation of the RSV model involves estimation of mean parameters $\{\alpha_0, \alpha_1\}$, variance parameters $\{\beta, \gamma, \sigma_\eta, \phi_1\}$, and the transition probability parameters $\{p_{01}, p_{10}\}$, where p_{ij} represents the transition probability of going from state i to j.

In summary, the RSV model specification combines a level effect and a conditional volatility process that captures all possible exogenous shocks. Note that the RSV model reduces to the SSV model when there is no regime shift in the data -i.e., when γ is restricted to zero.

A closely related paper is So, Lam and Li (1998), which uses a switching stochastic volatility model to explain the persistence in the log volatility for S&P 500 index weekly returns. Using a three-state model (with high, medium and low volatility states), So et al. (1998) find that the volatility state is less persistent, while the low volatility state is more persistent. Our model slightly differs from So et al. (1998). In our RSV model, the drift term of the conditional variance is a function of both current and last period states, while in So et al. (1998) the conditional variance is a function only of the current period state. This difference in our volatility specifications also leads to differences in our likelihood functions and, hence, in our posterior densities.

Estimation of the RSV model involves estimating two latent variables -i.e., h_t and s_t in addition to the model parameters. In the presence of two latent variables, the likelihood function for the model needs to be integrated over all the possible states of the two latent variables. Jacquier, Polson and Rossi (1993) show that maximum likelihood based methods tend to fail under complex specifications of the likelihood function. Consequently, we resort to Monte Carlo Markov Chain (MCMC) methods to estimate the

[2] For example, Cox, Ingersol and Ross (1985) and Anderson and Lund (1997).

RSV model.[3] Note that Model 3 can easily incorporate more complicated dynamics. For example, we can make σ_η a function of a latent state, or we can specify an ARMA structure for the conditional mean equation (with parameters driven by the latent state), or we can consider multiple regimes. Our estimation technique can accommodate all these extensions.

III. Data

The data consists of annualized yields based on weekly observations of three-month U.S. T-bill data for the period 01/06/60 to 06/03/98 (2003 weekly observations) and is obtained from the Chicago Federal Reserve's database. Wednesday's rates are used and if Wednesday is a trading holiday, then, Tuesday's rates are used. Similar data sets have been previously used by Anderson and Lund (1997) and Gray (1996).

Figure 1 plots the annualized yields and also the first differences in yields based on weekly observations of three-month T-Bill data. There are several episodes of large fluctuations in nominal interest rates: the oil shocks during 1969-71 and 1973-75, the Federal Reserve monetary experiment of 1978-82, and the market crash of 1987. This observation suggests more than a single regime in the data. For instance, we can think of a high volatility regime during the periods cited above and a low volatility regime during rest of the periods. We could allow for an additional regime, as in Hamilton and Susmel (1994), to capture outliers in the data. Following the existing switching literature, however, we limit ourselves to two regimes for the underlying volatility.

The weekly three-month T-Bill rate (r_t) is non-stationary[4]. First differencing makes the data stationary. Using Box-Jenkins methods, an ARIMA(1,1,0) model seems to provide a satisfactory fit for the autocorrelations in the yields. Following Pagan and Schwert (1990) and Ball and Torous (1995, 1999), we fit the RSV model (2) to the residuals (RES_t) from regressing Dr_t on a constant and Dr_{t-1}. For the purpose of estimation and comparison to alternative volatility models, we write our mean adjusted version of the RSV model as

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[3] Appendix A gives the details of our MCMC estimation and Appendix B presents the results of a simulation experiment using our estimation method.

[4] ADF tests could not reject the null of a unit root in the yield series.

$$
\begin{aligned}
&\Delta r_t - (\hat{a}_0 + \hat{a}_1 \Delta r_{t-1}) \equiv RES_t \\
&RES_t = \sqrt{h_t r_{t-1}^{2a}}\, e_t, \qquad a = 0.5 \\
&\left(\ln(h_t) - m_{s_t}\right) = f_1\left(\ln(h_{t-1}) - m_{s_{t-1}}\right) + \sqrt{s_h^2}\; h_{t-1} \\
&m_{s_t} = b + g s_t \qquad g > 0 \quad s_t = \{1,2\} \qquad\qquad (3)
\end{aligned}
$$

where all the assumptions on the error terms made in (2) still hold. Again, note that when we set γ to zero, the RSV model reduces to the SSV model.

Table 1 presents the summary statistics of the data. Changes in yields, Δr_t, seem to be left skewed indicating that periods of high T-Bill returns were less common compared to periods with low returns. There is also a strong evidence of kurtosis in the return series. The Ljung-Box statistic suggests that there is a high degree of autocorrelation for the raw yields (r_t). There is a very high persistence in the raw series. On the other hand, Δr_t series seems to be much less persistent and is characterized by low autocorrelations. High autocorrelation in the two series $(\Delta r_t)^2$ and $\log(\Delta r_t)^2$ suggests the kind of non-linearity in the data that can be explained by a SV model. Table 2 presents the results from GARCH tests on the data. We report the Ljung-Box statistic for the squared residuals (RES_t^2) at various lags. The null of no GARCH effects is strongly rejected by the data.

IV. Results from the Stochastic Volatility models

To benchmark our results, first, we ignore the possibility of regime-switching in the data. The results from the MCMC estimation of the SSV model are presented in Table 3, where the parameter set is $\theta = \{\beta, \phi, \sigma_\eta\}$. The persistence parameter ϕ is very high indicating that the half-life of a volatility shock, measured as $-\ln(2)/\ln(\phi)$, is about fourteen weeks. Standard errors for the parameters are small indicating that parameters are highly significant. Figure 2 plots the posterior densities of the parameters. All the parameters have symmetric densities while half-life density is right skewed indicating that half-lives longer than fourteen weeks are more common.

We, next, estimate the RSV model for our weekly interest data set. Table 5 presents the prior and posterior parameter estimates of the parameter set θ in our model, where $\theta = \{\beta, \gamma, \phi, \sigma_\eta, p_{01}, p_{10}\}$. Standard errors for the parameters are small as before.

The persistence parameter, ϕ, drops significantly to 0.628 from 0.951 in the SSV model. This implies that a switch in the latent regime creates a high persistence in volatility and confirms the earlier results of Hamilton and Susmel (1994), Cai (1994) and So, Lam and Li (1998). The distribution of ϕ is left skewed with a median 0.647 (see Figure 3), implying that even lower persistence than 0.628 is common. The transition probabilities, p_{00} and p_{11}, are estimated as 0.994 and 0.966. These estimates are comparable to 0.9896 and 0.9739 respectively reported in Gray (1996) and 0.9878 and 0.9402 respectively reported in Cai (1994). Our results imply that the effect of a volatility shock is much more persistent in the low volatility state than in the high volatility state. A volatility shock lasts on average of, at least, 100 weeks in the low volatility state compared to about 30 weeks in the high volatility state, where duration of the shock is obtained as $(1-p_{ii})^{-1}$. Figure 3 plots the Gaussian kernel densities for the posterior parameter estimates. The posterior densities seem to be symmetric for β and γ and right skewed for σ_η (with a median 0.897). Figure 4 plots the Gibbs parameter estimates from 1200 runs (details in appendix A). Gibbs runs indicate no autocorrealtion in successive draws. Figure 5 plots autocorrelations for the parameters. The autocorrelations become insignificant at very early lags implying that the Gibbs draws are drawn at random.

Tables 4 and 6 present the correlations between the parameters. Both Tables 4 and 6 report strong negative correlation between β and ϕ, and ϕ and σ_η. Table 6 also reports strong positive correlations between β and γ, and β and σ_η. Together, these results imply that 1) as the variance persistence decreases the unconditional variance -i.e., the long-run mean of $\ln(h_t)$- increases, 2) large volatility shocks are not as persistent as small volatility shocks and 3) large volatility shocks tend to be associated with higher long-run mean compared to small volatility shocks.

The first two panels of Figure 6 plot the T-Bill yields and the residuals from a regression of Dr_t on a constant and Dr_{t-1}, respectively. The third panel plots the underlying annualized volatility (generated by a multi-move simulation smoother), and the fourth panel plots the simulated smoother probabilities of being in high volatility state, i.e., $Prob(s_t = 1)$. Following Hamilton (1988), we consider an observation as belonging to state one if the smoothed probability is higher than 0.5. The simulation smoother shows periods of high volatility during the oil shocks of 1969 and 1973, the 1979-83 Federal Reserve monetary experiment, and the market crash of 1987. The smoother probabilities indicate that there is a large probability that the T-Bill yields during 1969, 1973, 1979-82, and 1987-88 belong to a high volatility regime. This dating is in agreement with the dating reported by Cai (1994) and Gray (1996).

V. Performance of the RSV Model

We conduct an extensive evaluation of the in-sample and out-of-sample performance of the SV two-factor models and other two-factor models, based on the GARCH family of models. We consider three popular GARCH models: GARCH(1,1) model, GARCH(1,1)-L model -i.e., GARCH(1,1) with an asymmetry effect of negative lagged errors, to capture the leverage effect- and EGARCH(1,1) model. The first GARCH model is the formulation used by Longstaff and Schwartz (1992). The second and third GARCH models retain a leverage effect, as in Brenner et al. (1996). All the GARCH models are specified to include a level effect (for specifications see Table 7). The MLE results for the three GARCH models are presented in Table 7. There is evidence for a leverage effect based on the significant t-statistic for κ in the GARCH(1,1)-L model and the significant t-statistic for δ_2 in the EGARCH(1,1) model. The leverage effect, however, is small relative to the usual size found in equity returns. All the estimates in the conditional variance equation are significant for the three models. Note that the estimates show the usual high persistence in the conditional variance.

We extract one-week(step)-ahead in-sample forecast variances from the single state and regime-switching two-factor models and compare them to other models. In addition to the full sample period, 01/06/60-06/03/98, we consider three sub-sample periods. The three sub-sample periods are: (1) 01/06/60-31/12/78, (2) 01/06/60-31/12/82, and (3) 01/06/60-31/12/91. The first sample includes the oil shocks, the second sample includes the Fed monetarist experiment of 1979-82, and the third sample includes the October1987 stock market crash. Based on the estimates for the three sub-samples, we estimate out-of-sample forecasts until the end of the sample. We also consider shorter samples and shorter out-of-sample forecast periods. As an example, we include a fourth sample 01/01/76-31/12/87, which allows an evaluation of the performance of the model in a shorter data set. This forth sample has two well defined spells of high volatility: the Fed experiment and the October 1987 stock market crash.

Figures 7 and 8 show the in-sample (annualized) conditional volatilities implied by all the models. The conditional volatilities from two-factor models are relatively less smooth compared to those from the GARCH type models. This is because the two-factor models are more sensitive to shocks. For example, the RSV two-factor model picks up an outlier in late 1982, which goes undetected by the other models.

Table 8 shows the likelihood function for all the models. The RSV has the biggest likelihood. Unfortunately, the GARCH models and the SSV models are not nested. Therefore, standard likelihood ratio tests are not correct. In addition, standard likelihood ratios cannot be used for the SSV model and the RSV model since there are unidentified parameters under the null hypothesis of no-switching -see Hansen (1992). Therefore, Table 8 shows four different in-sample evaluation criteria for the different models. The stochastic volatility models perform better than all the GARCH models and the SSV model. In particular, the RSV model has a higher likelihood function, higher adjusted R^2, and higher AIC/SBC values relative to the GARCH models and the SV model. Table 8 also reports posterior odds ratios of the competing model with respect to the constant variance model –see Kim and Kon (1994). If the odds ratio is positive, then the competing model is "more likely" to have generated the data than the constant variance model. The model with the highest value of posterior odds ratio represents the "most likely" competing model specification. The stochastic volatility models have higher odds ratios than GARCH models. In particular, the RSV model has an odds ratio at least 56% higher than the other competing models. Among the two-factor GARCH models, with the exception of the Adjusted R^2 criteria, the E-GARCH(1,1) performs better than the other models for all the evaluation measures. The E-GARCH(1,1) is also the model used by Ball and Torous (1999) to evaluate the in-sample evaluation of the SSV model. Based on these considerations, the E-GARCH(1,1) is the GARCH model we select to evaluate the out-of-sample performance of the SV models.

Table 9 presents in-sample and out-of-sample one-step ahead forecasts for all the models for the four different sub-samples. We present the mean squared errors (MSE) and mean absolute errors (MAE) for the SSV model, the RSV model, a constant volatility model, and for the best performing GARCH model, the E-GARCH(1,1). We keep a constant volatility model in our out-of-sample comparison, given the results in Figlewski (1997), where the constant volatility performs well relative to GARCH models. Table 9 shows that the RSV model tends to outperform the GARCH and SSV models. Consistent with the in-sample results of Table 8, the RSV model always beats in-sample the other formulations. Out-of-sample, the RSV tends to do better than the EGARCH and SSV model. The out-of-sample performance of the RSV model, however, is similar to the out-of-sample performance of the SSV model. Consistent with Figlewski (1997), the constant variance model shows a good out-of-sample performance, especially in the MSE metric. Note that the constant variance model in the first sub-sample beats all the other models. The E-GARCH model never performs better than the SV models. The last sub-sample presents a short period of out-of-sample forecasts, only one year. Again, the RSV model is the dominating model.[5]

VI. Conclusions

In this paper, we introduce regime-switching in a stochastic volatility model to explain the behavior of short-term interest rates. The regime-switching stochastic volatility process for interest rates captures all possible exogenous shocks that are either continuous in the form of `market-news' events or discrete as occurring from possible changes in underlying regime. We introduce the regime-switching stochastic volatility process in a two-factor model for the short-term interest rate. We estimate the two-factor model using a Gibbs Sampling based Markov Chain Monte Carlo algorithm that is robust to the usual non-linearities in the likelihood function. We find that the usual high volatility persistence is substantially reduced by the introduction of regime-switching. We conduct an extensive in-sample and out-of-sample evaluation of several two-factor models. We use several sub-samples and different evaluation criteria to compare the RSV model with other GARCH models and single-state stochastic volatility model. Overall, our results are very supportive of our RSV two-factor model.

[5] For the fourth sample, we also calculate (not reported) out-of-sample forecasts for a two-year period and a ten-year period. Overall, the results are similar, although as the out-of-sample forecasting period is extended, the performance of the SSV model becomes very similar to the performance of the RSV model.

In-sample and out-of-sample, the RSV model tends to outperform all the other two-factor models.

Appendix A: **The Gibbs Algorithm for Estimating RSV Model**

In the RSV model (2), we need to estimate the parameter vector θ ={β, γ, σ_η, ϕ_1, p_{01}, p_{10}} along with the two latent variables H_t = {h_1,...,h_t} and S_t ={s_1,….,s_t}. The parameter set therefore consists of ω = {H_t, S_t, θ} for all t. We use Bayes theorem to decompose the joint posterior density as follows.

$$f(H_n, S_n, q) \propto f(Y_n|H_n)\, f(H_n|S_n, q)\, f(S_n|q)\, f(q)$$

We next draw the marginals f(H_t| Y_t, S_t ,θ), f(S_t|Y_t,H_t,θ) and f(θ|Y_t, H_t S_t), using the Gibbs sampling algorithm described below:

Step 1:

Specify initial values $q^{(0)} = \{b_1^{(0)}, g^{(0)}, s_h,^{(0)}, f^{(0)}, p_{01}^{(0)}, p_{10}^{(0)}\}$. Set i =1.

Step 2:

Draw the underlying volatility using the multimove simulation sampler of De Jong and Shephard (1995), based on parameter values from step 1. The underlying volatility vector for all the data points is obtained as a function of underlying disturbances that are drawn as a block using a simulation smoother. Consider the RSV model (3), reproduced below:

$$
\begin{aligned}
&\Delta r_t - (\hat{a}_0 + \hat{a}_1 \Delta r_{t-1}) \equiv RES_t \\
&RES_t = \sqrt{h_t r_{t-1}^{2a}}\, e_t, \qquad a = 0.5 \\
&\left(\ln(h_t) - m_{s_t}\right) = f_1\left(\ln(h_{t-1}) - m_{s_{t-1}}\right) + \sqrt{s_h^2}\; h_{t-1} \\
&m_{s_t} = b + g s_t \qquad g > 0 \quad s_t = \{1,2\}
\end{aligned}
\tag{3}
$$

The conditional mean equation can be written as,

$$\ln(RES_t^2) = \ln(h_t) + \ln(r_{t-1}) + \ln(e_t^2) \tag{A -1}$$

The term $\ln(\varepsilon_t^2)$ can be approximated by a mixture of seven normal variates (Chib, Shephard, and Kim (1998)).

$$
\begin{aligned}
&\ln(e_t^2) = z_t \\
&f(z_t) = \sum_{i=1}^{7} f_N\left(z_i \middle| m_i - 1.2704, v_i^2\right) \quad i = \{1,2,....7\}
\end{aligned}
\tag{A - 2}
$$

Now, (A-1) can be written as

$$\ln(RES_t^2) = \ln(h_t) + \ln(r_{t-1}) + \left[z_t \middle| k_t = i\right] \qquad \text{(A - 3)}$$

where k_t is one of the seven underlying densities that generates z_t. Once the underlying densities k_t, for all t, are known, (A-3) becomes a deterministic linear equation and along with the RSV model (3) can be represented in a linear state space model. Next, apply the De Jong and Shephard (1995) simulation smoother to extract the underlying log volatility from the observed data.

Step 3:

Based the on output from steps 1 and 2, the underlying k_t in (A-3) is sampled from normal distribution as follows -see Chib, Shephard and Kim (1998):

$$f\left[z_{t=i} \middle| \ln(y_t^2), \ln(h_t)\right] \propto q_i f_N\left(z_i \middle| \ln(h_t) + m_i - 1.2704, v_i^2\right) \qquad i \leq k \qquad \text{(A - 4)}$$

For every observation t, we draw the normal density from each of the seven normal distributions $\{k_t = 1,2,..,7\}$. Then, we select a "k" based on draws from uniform distribution.

Step 4:

Based on the output from steps 1, 2 and 3, we draw the underlying Markov-state following Carter and Kohn (1994). We use the smoother for the above state-space model (3), to derive the vector of underlying state variable s_t, t = 1,2,...,n

Step 5:

Cycle through the conditionals of parameter vector $\theta = \{\beta, \gamma, \sigma_\eta, \phi_1, p_{01}, p_{10}\}$ for the volatility equation using Chib (1993), using output from steps 1-4. Assuming that f (θ) can be decomposed as:

$$f(q|Y_n, H_n, S_n) \propto f(b|Y_n, H_n, S_n, q_{-b})\, f(g|Y_n, H_n, S_n, q_{-g})\, f(s^2|Y_n, H_n, S_n, q_{-s^2})$$
$$f(f|Y_n, H_n, S_n, q_{-f})\, f(p_{01}, p_{10}|Y_n, H_n, S_n, q_{-p_{ij}}) \qquad \text{(A -5)}$$

where θ_{-j} refers to the θ parameters excluding the jth parameter. The respective conditional distributions (normal for β, γ and ϕ, inverse gamma for σ^2 and beta for p_{ij}) are described in Chib (1993). The parameter γ is drawn using an inverse CDF with the restriction that it is positive. The prior means and standard deviations are specified in Tables 3 and 5.

Step 6: Go to step 2.

Estimation of SSV model (2) has the same steps as in RSV model (3), except that we do not have to draw the latent states and transition probabilities. For the Gibbs estimation, we leave out the first 4000 draws (i.e., burn-in iterations are 4000) and sample from the next 6000 draws. We choose every fifth observation to minimize, and if possible eliminate, any possible correlation in the draws. Our effective number of draws therefore drops to 1200 (i.e., effective test iterations are 1200). We construct 95% confidence intervals for the parameters, based on 1200 draws. We construct the standard errors for the parameters using the batch-means method -see Chib (1993). We estimate the density functions for the parameters using the Gaussian kernel estimator -see Silverman (1986).

Appendix B: Monte Carlo Experiment with the Gibbs Algorithm

We perform a Monte Carlo experiment of the RSV model (3), without level effects. That is:

$$RES_t = \sqrt{h_t}\,e_t$$
$$\left(\ln(h_t) - m_{s_t}\right) = f_1\left(\ln(h_{t-1}) - m_{s_{t-1}}\right) + \sqrt{s_h^2}\; h_{t-1}$$
$$m_{s_t} = b + g s_t \qquad g > 0 \quad s_t = \{1,2\}$$

Using the transition probabilities p_{01} and p_{01} , we generate a state vector (with values 0 or 1) of size 1000. Using the state vector and the "true" parameters β, γ, $\sigma_{\eta,}$ and ϕ_1, we generate stochastic volatility, $\ln(h_t)$. Based on the above model, the stochastic volatility series is used it to generate the residual vector, RES_t. (All the true parameter values used in the simulation are listed below.) Then, taking RES_t as given, we estimate the parameter set $\theta = \{\beta, \gamma, \sigma_\eta, \phi_1, p_{01}, p_{10}\}$ using the MCMC algorithm as explained in appendix A. We set the number of burn–in iterations equal to 4000 and the number of effective test iterations equal to 1200. Thus, we construct the 95% confidence intervals for the parameters based on 1200 draws. We construct the standard errors for the parameters using the batch-means method -see, Chib (1993). The results are reported in Table B.1.

Table B.1

Results from a Monte Carlo experiment

T (sample size) :1000

parameter	True values	Prior Values		Posterior Values		
		Mean	Std. deviation	Mean (std. Error)	Std. deviation	95% Confidence Interval
β	0.7	0	50	0.759 (0.003)	0.112	(0.500- 0.986)
γ	1.5	1	50	1.477 (0.018)	0.191	(1.051 -1.819)
ϕ	0.4	0	1	0.391 (0.005)	0.085	(0.202- 0.054)
σ^2	0.6	-	-	0.733 (0.005)	0.109	(0.538 - 0.976)
p_{01}	0.01	0.2	0.16	0.011 (0.000)	0.007	(0.003-0.035)
p_{10}	0.04	0.2	0.16	0.074 (0.003)	0.036	(0.026-0.166)

*Prior distribution of σ^2 (inverse gamma) is improper

We find that the posterior means of parameters are quite close to the true values. The standard errors are small, indicating a high precision of the posterior means. For the variance and the transition probability p_{10}, the posterior means are slightly higher than true values. However, they clearly lie within the 95% confidence bounds.

Figure B.1 shows the latent volatility and states. The top panel consists of simulated residuals RES_t. The second panel presents both the true and latent volatility, the latter obtained using the simulation smoother. The third panel presents the true states –i.e., either 0 or 1- and smoother probabilities of being in the high volatility state. From the second and third panels, we see that the smoother volatility and probabilities closely approximate their true counterparts.

Figure B.1. Simulated Yields and Corresponding Latent Volatility and States

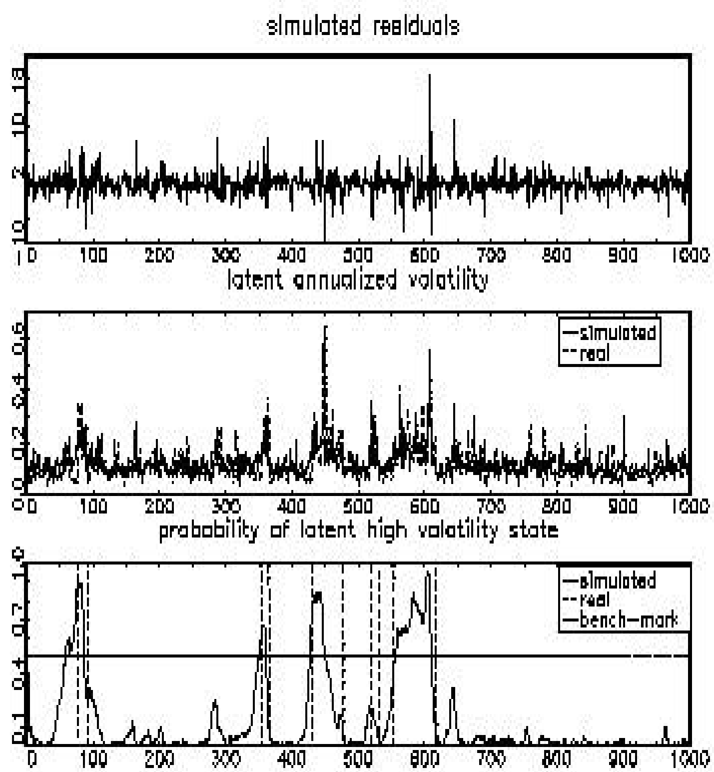

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Table 1

Summary statistics for weekly interest rates on 3-month T-Bills for the period 1/06/60 to 06/03/98

	r_t	Δr_t	$(\Delta r_t)^2$	$\log (\Delta r_t)^2$
Mean	6.0436	0.00018962	0.048828	-5.6721
Standard error	0.060171	0.0049374	0.0044547	0.057084
Variance	7.2591	0.048852	0.039769	6.5302
Standard error	0.31925	0.0044540	0.0083140	0.20325
Skewness	1.2504	-1.0377	8.3522	-0.25630
Standard error	2.0700	0.0055073	0.019342	0.71194
Kurtosis	4.8781	17.658	88.586	2.9414
Standard error	22.157	0.0086464	0.049010	6.6579
Ljung-Box (24)	1733.4	12.018	100.96	190.78

Notes:

- Ljung-Box (24). Ljung-Box statistics calculated with 24 lags. $\chi^2_{(24)}$ critical value for a 95% confidence level is 36.4.

Table 2

Tests for GARCH effects in weekly interest rates on 3-month T-Bills for the period 1/06/60 to 06/03/98

Lag	Ljung-Box Statistic	$\chi^2_{(lag)}$ statistic (95% confidence level)
6	186.68	12.6
12	137.30	21
18	109.40	28.9
24	104.47	36.4
36	104.79	55

Notes:

- We obtain the residuals (RES_t) from regressing Dr_t on a constant and Dr_{t-1} and report the Ljung-Box statistics for the squared residuals at different lags. The Ljung-Box statistic for squared residuals is highly significant at all lags.

Table 3

Results from the MCMC estimation of the Single-State Stochastic Volatility (SSV) model using weekly 3-month T-Bill yields for the period 01/06/60 to 06/03/98

parameter	Prior Values		Posterior Values		
	Mean	Standard Deviation	Mean (std. error)	Standard Deviation	95% Confidence Interval
β	0.05	1	2.831 (0.020)	0.219	(2.366- 3.235)
ϕ	0	10	0.951 (0.000)	0.009	(0.932- 0.969)
σ^2	-	-	0.190 (0.002)	0.023	(0.150- 0.241

<u>Notes:</u>

- The SSV model used in Table 1 (Model 1):

  $\Delta r_t - (\hat{a}_0 + \hat{a}_1 \Delta r_{t-1}) \equiv RES_t$

  $RES_t = \sqrt{h_t r_{t-1}^{2a}}\, e_t, \qquad a = 0.5$

  $(\ln(h_t) - m) = f_1 (\ln(h_{t-1}) - m) + \sqrt{s_h^2}\; h_{t-1}$

  $m = b$

- The sample size is 2003. Prior distribution of σ^2 (inverse gamma) is improper. Details about the model estimation are in appendix A.

Table 4

Correlation matrix of the parameters for the SSV model using weekly 3-month T-Bill yields for the period 01/06/60 to 06/03/98

Parameters	β	σ^2	ϕ
β	1	0.030	-0.202
σ^2	0.030	1	-0.551
ϕ	-0.202	-0.551	1

Table 5

Results from MCMC estimation of the Regime-switching Stochastic Volatility (RSV) model using weekly 3-month T-Bill yields for the period 01/06/60 to 06/03/98

Parameters	Prior Values		Posterior Values		
	Mean	Standard Deviation	Mean (std error)	Standard Deviation	95% Confidence Interval
β	0	50	2.580 (0.001)	0.098	(2.378- 2.769)
γ	1	50	2.746 (0.022)	0.247	(2.258 -3.220)
ϕ	0	1	0.628 (0.001)	0.046	(0.526- 0.708)
σ^2	-	-	0.931 (0.002)	0.123	(0.726-1.207)
p_{01}	0.2	0.16	0.006 (0.001)	0.003	(0.002- 0.013)
p_{10}	0.2	0.16	0.034 (0.001)	0.013	(0.014- 0.063)

Notes:

- The RSVmodel is estimated in Table 5 (Model 3):

$$\Delta r_t - (\hat{a}_0 + \hat{a}_1 \Delta r_{t-1}) \equiv RES_t$$

$$RES_t = \sqrt{h_t r_{t-1}^{2a}}\, e_t, \qquad a = 0.5$$

$$\left(\ln(h_t) - m_{s_t}\right) = f_1\left(\ln(h_{t-1}) - m_{s_{t-1}}\right) + \sqrt{s_h^2}\; h_{t-1}$$

$$m_{s_t} = b + g s_t \qquad g > 0 \quad s_t = \{1,2\}$$

- The sample size is 2003. Prior distribution of σ^2 (inverse gamma) is improper. Details about the model estimation are in appendix A.

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Table 6

Correlation matrix of the parameters for the RSV model using weekly 3-month T-Bill yields for the period 01/06/60 to 06/03/98

parameters	β	γ	σ^2	ϕ	a :p_{01}	B: p_{10}
β	1.000	0.606	0.501	-0.530	0.077	0.170
γ	0.606	1.000	0.424	-0.373	-0.179	-0.105
σ^2	0.501	0.424	1.000	-0.766	0.236	0.340
ϕ	-0.530	-0.373	-0.766	1.000	-0.240	-0.382
a :p_{01}	0.077	-0.179	0.236	-0.240	1	0.414
b: p_{10}	0.170	-0.105	0.340	-0.382	0.414	1

Table 7

Results from MLE estimation of GARCH models using weekly 3-month T-Bill yields for the period 01/06/60 to 06/03/98

	α_0	κ	δ_1	δ_2	α_1	β_1
GARCH(1,1)	0.7763	-	-	-	0.0231	0.8514
	(5.110)	-	-	-	(7.164)	(47.012)
GARCH(1,1)-L	0.7866	0.0094	-	-	0.0192	0.8469
	(5.150)	(2.566)	-	-	(6.5007)	(46.993)
EGARCH(1,1)	0.0536	-	0.1441	-0.0193	-	0.9441
	(2.8021)	-	(11.5671)	(-3.4341)	-	(128.91)

Notes:

- t-statistics are reported in parenthesis.
- Model specifications:

GARCH(1,1) with level effect:

$$\Delta r_t - (\hat{a}_0 + \hat{a}_1 r_{t-1}) \equiv RES_t$$

$$RES_t = \sqrt{h_t r_{t-1}^{2a}}\, e_t, \qquad a = 0.5 \qquad (e_t \mid \Omega_{t-1}) \sim N(0,1)$$

$$h_t = a_0 + a_1 u_{t-1}^2 + b_1 h_{t-1} \qquad t > 1$$

$$h_1 = \frac{a_0}{1 - a_1 - b_1} \qquad t = 1$$

GARCH(1,1)-L with level effect :

$$\Delta r_t - (\hat{a}_0 + \hat{a}_1 r_{t-1}) \equiv RES_t$$

$$RES_t = \sqrt{h_t r_{t-1}^{2a}}\, e_t, \qquad a = 0.5 \quad (e_t \mid \Omega_{t-1}) \sim N(0,1)$$

$$h_t = a_0 + k d_{t-1} u_{t-1}^2 + a_1 u_{t-1}^2 + b_1 h_{t-1} \qquad t > 1$$

$$d_{t-1} = \begin{cases} 0 \text{ if } u_{t-1} > 0 \\ 1 \text{ if } u_{t-1} \le 0 \end{cases} \qquad h_1 = \frac{a_0}{1 - \frac{k}{2} - a_1 - b_1} \qquad t = 1$$

EGARCH(1,1) with levels effect:

$$\Delta r_t - (\hat{a}_0 + \hat{a}_1 r_{t-1}) \equiv RES_t$$

$$RES_t = \sqrt{h_t r_{t-1}^{2a}}\, e_t, \qquad a = 0.5 \quad (e_t \mid \Omega_{t-1}) \sim N(0,1)$$

$$\ln(h_t) = a_0 + d_1\left(|x_{t-1}| - \sqrt{\frac{2}{p}} \right) + d_2 x_{t-1} + b_1 \ln(h_{t-1}) \qquad t > 1$$

$$x_{t-1} = \frac{e_t}{\sqrt{h_t}} \qquad \ln(h_1) = \frac{a_0}{1 - b_1} \qquad t = 1$$

Table 8

In-sample comparison of alternative models for the entire sample period
01/06/60 to 06/03/98 (sample size: 2003)

	number of parameters	Log-Likelihood	AIC	SBC	Adj R^2	Odds ratios
Constant Variance	1	-8589.23	-8590.23	-8593.03	-0.477	-
GARCH(1,1)	3	-7965.81	-7968.81	-7977.22	0.222	623.42
GARCH(1,1)-L	4	-7962.05	-7966.05	-7977.25	0.205	627.18
EGARCH(1,1)	4	-7953.13	-7957.13	-7968.34	0.220	636.1
SSV model	3	-7825.64	-7828.64	-7837.04	0.405	763.59
RSV model	6	-7397.39	-7401.39	-7412.60	0.602	1191.84

Notes:

- AIC: log-likelihood value *less* number of parameters
- SBC: log-likelihood value *less* 0.5 log (T*number of parameters)
  where T: sample size
- MSE: $\dfrac{\sum_{t=1}^{T}\left[RES_t^2 - h_t\right]^2}{T}$
- MAE: $\dfrac{\sum_{t=1}^{T}\left|RES_t^2 - h_t\right|}{T}$
- Adj R^2: Adjusted R^2 is calculated for the regression $Res_t^2 = a + bh_t + u_t \quad u_t \sim N(0,1)$ {t= 1,…..N}, where RES_t are the residuals from regressing Dr_t against constant and Dr_{t-1} and h_t $\{t = 1,\ldots N\}$ are conditional volatility estimates
- Odds ratio: the posterior odds ratio of alternative specifications relative to the constant variance. This is obtained as difference of the Schwartz Bayesian Criterion (SBC) of each competing model and the SBC of the constant variance model –see, Kim and Kon (1994).
- All the models used here are described in Tables 3-7.

Table 9

In-sample and out-of-sample comparison of alternative models for three different sample periods

Sample 1					
	In-sample (T: 990) 01/06/60-31/12/78			Out-of-sample (T:1012) 01/01/79-06/03/98	
	MSE	MAE	Adj. R^2	MSE	MAE
Const. Variance	0.0109	0.0266	-0.0495	0.0702	0.0682
EGARCH(1,1)	0.0104	0.0255	0.104	0.0707	0.0694
SSV model	0.0104	0.0255	0.409	0.0706	0.0688
RSV model	0.0099	0.0243	0.615	0.0705	0.0685

Sample 2					
	In-sample (T: 1199) 01/06/60-31/12/82			Out-of-sample (T:803) 01/01/83-06/03/98	
	MSE	MAE	Adj. R^2	MSE	MAE
Const. Variance	0.0661	0.0714	-0.0671	0.0025	0.0132
EGARCH(1,1)	0.0621	0.0679	0.210	0.0027	0.0130
SSV model	0.0628	0.0680	0.485	0.0026	0.0126
RSV model	0.0616	0.0668	0.612	0.0026	0.0123

Sample 3					
	In-sample (T: 1668) 01/06/60-31/12/91			Out-of-sample (T:334) 01/01/92-06/03/98	
	MSE	MAE	Adj. R^2	MSE	MAE
Const. Variance	0.0489	0.0561	-0.0562	0.0002	0.0067
EGARCH(1,1)	0.0461	0.0538	0.212	0.0001	0.0049
SSV model	0.0464	0.0538	0.481	0.0001	0.0047
RSV model	0.0458	0.0532	0.603	0.0001	0.0047

Sample 4					
	In-sample (T: 626) 01/01/76-31/12/87			Out-of-sample (T:104) 01/01/88-31/12/89	
	MSE	MAE	Adj. R^2	MSE	MAE
Const. Variance	0.1133	0.1081	-0.0993	0.0003	0.0119
EGARCH(1,1)	0.1090	0.1054	0.221	0.0005	0.0128
SSV model	0.1082	0.1051	0.494	0.0005	0.0124
RSV model	0.1031	0.1014	0.554	0.0003	0.0115

Notes:

- T: refers to the sample size.
- The best model is highlighted.
- MSE: $$\frac{\sum_{t=1}^{T}\left[RES_t^2 - h_t\right]^2}{T}$$
- MAE: $$\frac{\sum_{t=1}^{T}\left|RES_t^2 - h_t\right|}{T}$$
- The estimated coefficients from the in-sample period are used to generate one-week (step) ahead conditional

volatility estimates for the out-of-sample period. One-step ahead conditional volatility forecasts are generated using the following equations (based on the GARCH models defined under Table 7 and the SV models 2 and 3):

<u>GARCH(1,1):</u>

$$s^{2}_{t+s|t} = w_0 + (a_1 + b_1)^{s-1}(s^{2}_{t+1|t} - w_0)$$

where

$$w_0 = \frac{a_0}{1 - a_1 - b_1}$$

<u>GARCH(1,1)-L:</u>

$$s^{2}_{t+s|t} = w_0 + (a_1 + b_1 + \frac{k}{2})^{s-1}(s^{2}_{t+1|t} - w_0)$$

where

$$w_0 = \frac{a_0}{1 - a_1 - b_1 - \frac{k}{2}}$$

<u>EGARCH(1,1):</u>

$$\ln(s^{2}_{t+s|t}) = w_0 + (b_1)^{s-1}(\ln(s^{2}_{t+1|t}) - w_0)$$

where

$$w_0 = \frac{a_0}{1 - b_1}$$

<u>SSV model:</u>

$$\ln(s^{2}_{t+s|t}) = m + (f_1)^{s-1}(\ln(s^{2}_{t+1|t}) - m)$$

<u>RSV model</u>**:**

$$\ln(s^{2}_{t+s|t}) = \left[\ln(s^{2}_{t+s|t}) \middle| pr(s_{t+s} = 0 \mid s_{t+s-1})\right] \times pr(s_{t+s} = 0 \mid s_{t+s-1}) + \left[\ln(s^{2}_{t+s|t}) \middle| pr(s_{t+s} = 1 \mid s_{t+s-1})\right] \times pr(s_{t+s} = 1 \mid s_{t+s-1})$$

Figure 1. Weekly 3-month T-Bill percentage yields
(Sample: 01/06/60 to 06/03/98)

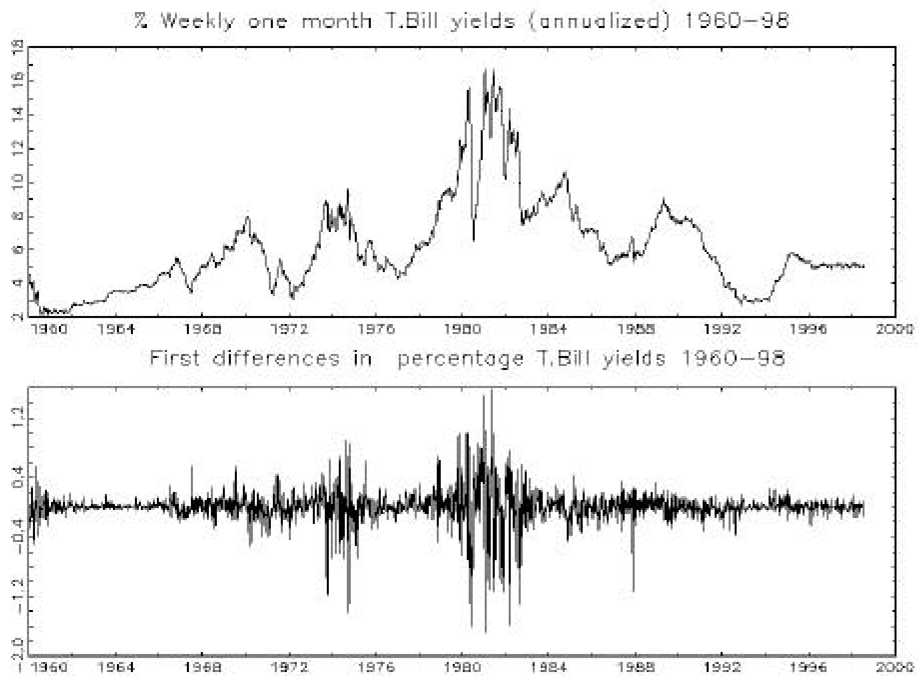

Figure 2. Posterior Density Plots for Parameters of the SSV Model

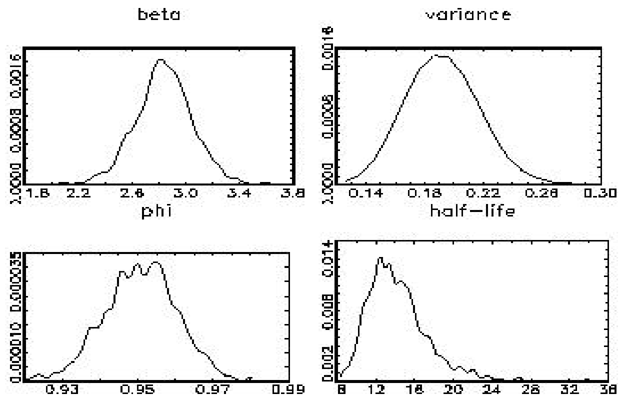

Figure 3. Posterior Density Plots for Parameters of the RSV model

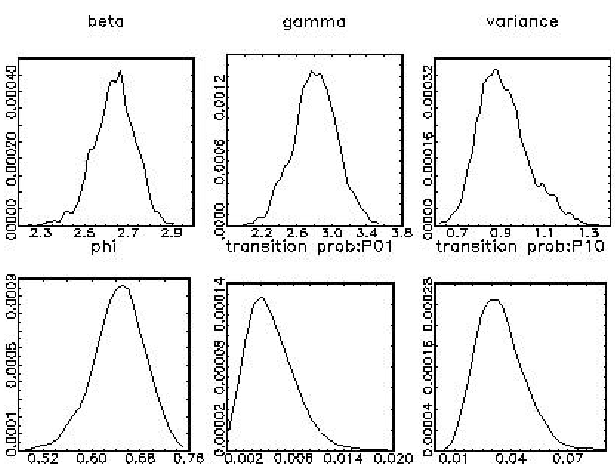

Figure 4. Gibbs Run for Parameters of the RSV Model

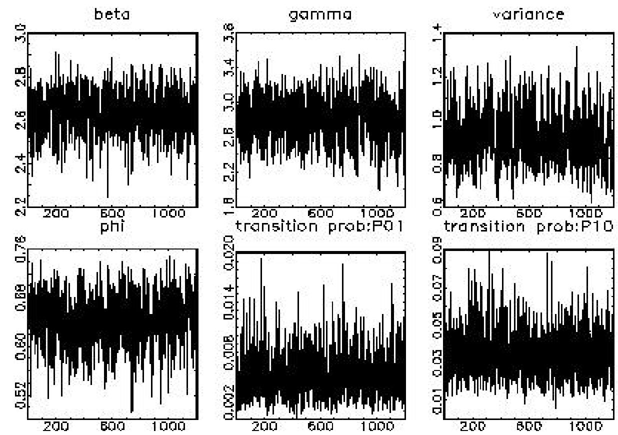

Figure 5. Autocorrelation Functions for Parameters of the RSV Model

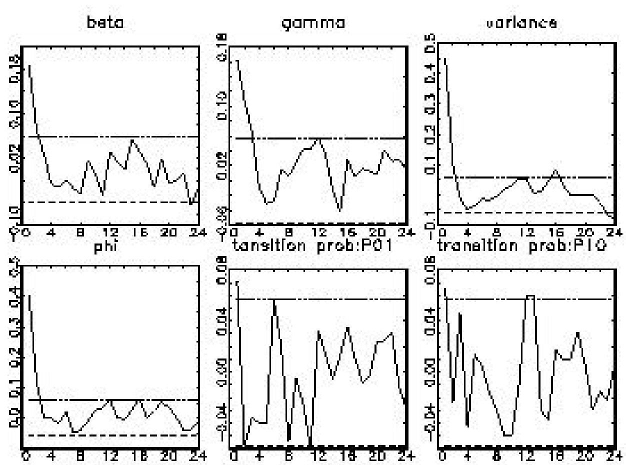

Figure 6. T-Bill Yields and Corresponding Latent Volatility and States
(Sample: 01/06/60 to 06/03/98)

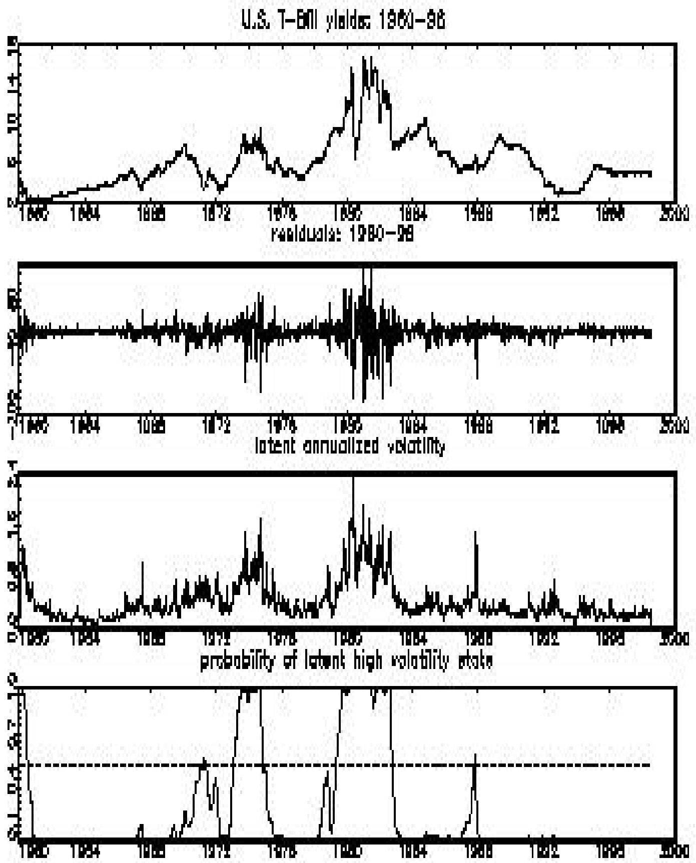

Figure 7. Comparison of in-sample Conditional Volatilities Across Different Models
(Sample: 01/06/60 to 06/03/98)