DIFFERENTIAL RATES OF RETURN

AND RESIDUAL INFORMATION SETS

A Discrete Approach

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Abstract

It is our purpose here to show the deep relationship between differential rates and their underlying information sets. To accomplish our task, we will make for the following stages: In the first place, we deal with scaled changes along a period and conditional rates of change within a discrete environment. Next, rings and algebras of sets are addressed, so as to provide information sets with a suitable structure and give grounds to differential rates. Afterwards, differential rates are presented rigourosly, and two important lemmas follow through: the first one makes possible the use of differential rates with restrictive assumptions on their information sets, as customary applications seem to require. The second lemma attempts a broader outcome in a general setting so as to cope with differential rates defined on more realistic information sets. Both lemmas contributerigorously to shape definitions of narrow and broad differential rates on residual information sets.

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01.- INTRODUCTION

It was since the 1970's that concerns about information sets gave rise to, at least, four cooperative and distinctive pathways of research and application:

a) For those who took up the Efficient Market Model, information sets $\mathbf{\Omega_t}$ became, as Fama (1970) put it: "a general symbol for whatever set of information is assumed to be fully reflected in the price [of a financial asset] ". Therefore, returns or prices assessment became contingent or conditional upon very general and pervading information sets. Nevertheless, no explicit effort was made in uncovering the structure underlying these sets, as if they were externally given to economic agents. A remarkable expansion on the Efficient Market Model is Elton-Gruber (1995).

b) A second track of enquiry followed from Fama's nurturing approach, while adding Black-Scholes and CAPM models. Researchers took advantage of advanced mathematics and build up a rigorous framework to cope with returns and prices valuation of financial assets. Information structures embedded in finite or infinite sample spaces were developed, drawing from sigma fields, probability theory and Ito's stochastic calculus (Ito, 1995). A survey on this successful attempt can be found in Dotham (1990). An updated survey on stochastic differential equations is given by Oksendal (1985).

c) Another departure was proposed by game theorists. As production of information comes accross by interacting with Nature, financial economists profitted indeed from this mindset, working on the underlying distribution functions of returns in financial assets. An outstanding introduction to the relationship between game theory and information can be followed in Rasmusen (1994).

d) A fourth pathway was tried by those researchers concerned with imperfect markets, asymmetric information, agency problems, transactions costs and other empirical issues. Financial economists, accountants, information scientists, econometricians, felt that many outcomes were at variance with some tenets claimed by models arising from the former pathways. On this account, Mayer (1992) is a worthy survey of the new approach. Although the insight of efficient markets was not contested, it was regarded as an extreme case of modelling. Furthermore, abstract information structures were felt as falling short of meaningful uses in contexts of application. Gonedes (1975, 1976) stated that "failure to explicitly consider the market for information may induce unwarranted inferences about the capital market". And Grossman and Stigler (1980) proved, in a particular case, the impossibility of informationally efficient markets. Not to be surprised, econometric models began to cope with limited information sets $\mathbf{\Lambda_t}$ (Cuthbertson,1996).

It is with the first and fourth pathways of research that this paper most closely holds eventually. In a previous work, Apreda (2000a), we presented a transaction costs approach to financial assets rates of return, whose main outcomes could be abridged this way:

- There are, at least, five broad types of transaction costs: those required by intermediation, microstructure, taxes, information and, lastly, the financial costs of handling transactions. Furthermore, those types of costs lead to an all-inclusive transaction costs function that can be embedded in a multiplicative model of differential rates.

- It is for each transaction cost type to fit in with an underlying information set which can be translated by a distinctive differential rate. In this way, nominal rates of return can be broken down into transaction cost components and a rate of return which is exclusive of transaction costs.

The purpose of this paper is to show how deep the relationship is between the differential rates and the underlying information sets. To accomplish our task, we will make for the following stages:

a) In the first place, we deal with scaled changes along a period and conditional rates of change.

b) Next, we address rings and algebras of sets, so as to provide information sets with a qualified framework and to give grounds to differential rates.

c) The main contribution of this paper is brought forward in section 06 where two lemmas are proved. The first one makes possible the use of differential rates, but with restrictive assumptions on their information sets, as the most customary applications seem to require. The second lemma attempts to get a broader outcome so as to cope with differential rates defined on more realistic information sets. Both lemmas contribute rigourosly to shape definitions of narrow and broad differential rates on residual information sets.

Remark:

The analysis will be kept within a discrete modelling framework lying on finite rings and algebras of sets, as it seems convenient in practice.

02.- SCALED CHANGES IN A PERIOD

Let **f(t)** be a measure of an economic or financial variable. For instance, stock or bonds prices, the pound sterling exchange rate, the performance of a certain level of economic activity, or the wheat price per bushel.

Formally, we are going to deal with functions

$$\mathbf{f} \; : \; \mathbf{I} \; \rightarrow \; \Re^{1} \qquad \text{where } \mathbf{I} \text{ is a compact subset in the set of real numbers } \Re^{1}$$

To measure the rate of change that **f(t)** undergoes along the period **[t ; T]** we can solve **h(t,T)** from the functional equation:

$$\mathbf{f(T) \; / \; f(t) \; = \; 1 + h(t,T)}$$

such that

$$\mathbf{h} \; : \; \mathbf{I \times J} \; \rightarrow \; \Re^{1} \qquad \text{where } \mathbf{I \times J} \text{ is a compact subset in } \Re^{2}$$

It follows that **h(t, T)** is positive when **f(T) ≥ f(t)**, negative otherwise. Solving for **h(t, T)**:

[01]

$$h(t,T) = [\ f(T) - f(t)\ ] \ /\ f(t) = \Delta f(t)\ /\ f(t)$$

and this adds to the absolute change of the variable scaled by **f(t)** on those points $p \in [\ t\ ;\ T\ ] \subseteq I$ where [01] is well defined.

Remarks:

i) Although regularity conditions may be included right now, we would rather keep the setting as simple as possible, leaving to each context of application the task of displaying either continuity or differentiability features in the extreme cases, or piece-wise linear or non linear functions in not so restrictive environments.

ii) Hubbard-Hubbard (1999) is an updated reference to nonlinear mathematics, while Berge (1997) focuses on topological spaces; both books will take the reader into the mainstream of mathematics as used in Financial and Economic Analysis. The best available introduction to complex economics dynamics seems Day (1994). Modelling prices and arbtirage in a nonlinear dynamics can be found in Apreda (1999a, 1999b).

As from now, and for ease of writing, when referring to some function in general terms, we will put a single dot instead of declaring all the independent variables. For example, when we write **f(.)**, it is because we make light of its particular structure.

02.01.- APPLICATION OF SCALED CHANGES TO FINANCIAL ECONOMICS

We need to qualify **f(.)** so as to make it suitable for Financial Economics purposes. Therefore, at moment " t " we have the price of a financial asset that comes up as a stock variable,

$$f(t) = P(t)$$

On the other hand, we deal at " T " with another stock variable, the price of the same financial asset at that moment, and a flow variable which provides income along **[t ; T]** , under the guise of dividends or interest payments:

$$f(T) = P(T) + I(t,T)$$

In this way, the scaled change in a period becomes the total rate of return of the financial asset.

[02]

$$r(t,T) = [\ P(T) + I(t,T) - P(t)\ ] \ /\ P(t)$$

In the following section, we will qualify this relationship as regards the underlying information sets which give sense to any assessment of final prices and flows.

Remarks:

The flow can be thought in a discrete setting (for instance, a coupon payment of interest taking place at certain point within the horizon [t, T], or in a continuous environment. When the duration of [t, T] is rather short, the flow may amount to zero. Some people claim, however, that even in this case expected accrued interest or dividends should be assessed.

03.- CONDITIONAL RATES OF CHANGE

Most of the time, we don't know the value of either **P(.)** or **I(.)** at date " **t** ". The only way to deal with [02] consists of substituting an estimated value for **P(T)** and **I(t , T)**, namely :

$$E[\,P(\,T\,)\,] \;+\; E[\,I(\,t\,,T\,)\,]$$

This last value is conditional on the economic agent's "information set" Ω_t. An information set means the set of all available information to him, up to the valuation date.

Remarks:

i) It has been customary to translate the phrase " all available information " within the realm of Fama's influential approach to efficiency in capital markets, by pointing out to weak, semi-strong or strong types of information sets (Fama 1970, 1991). An updated rendering is Elton-Gruber (1995). For a contesting approach, Shleifer's book (1999) and Daniel-Titman's paper (2000) are of interest.

ii) From the begininig of the 1970s, transaction costs theory (Williamson, 1996) and the incomplete contracts approach (Hart, 1995) have been stressing the role bounded rationality and opportunistic behaviour perform at impairing the quality of information sets. An attempt to make operational this effort can be found in Apreda (2000c).

Furthermore, [02] tells us that we assess not only **P(T) + I(t , T)** , but also **r(t , T)**. That is to say, the working relationship becomes:

[03]

$$<E[\,P(\,T\,)\,,\Omega_t\,] \;+\; E[\,I(\,t\,,T,\Omega_t\,)\,]> \;/\; P(\,t\,) \;=\; 1 \;+\; E\,[\,r(\,t\,,T\,,\Omega_t)\,]$$

In this way, rates of change carry on a conditional feature upon future states of the world. It is frequent to find the following format when dealing with the expectations operator:

$$E\,[\,r(\,t\,,T\,)\mid \Omega_t\,]$$

but we will give a strong reason on page 13 (see remarks iii) and iv)) to depart from the current usage.

As from now, we are going to avoid the symbol of the expectations operator **E[.]** , unless it were essential to do so, leaving to context the task of underscoring the operator or not. In fact, the dated information set should prevent us from any misunderstanding. For instance, if we wanted an ex~post valuation of [02], at the moment " T ", we would only need to write:

$$<\;P(\,T\,)\;+\;I(\,t\,,T\,)\;>\;/\;P(\,t\,)\;=\;1\;+\;r(\,t\,,T\,,\Omega_T)$$

Thus, the dated information set comes down to an actual marker for both ex~ante and ex~post assessments. On the other hand, the different measures of both ex~ante and ex~post assessments of financial assets rates of return can be attempted because we take stock on different information sets

$$\Omega_t \quad , \quad \Omega_T$$

Contrasting both information sets should be a sound practice, because most of the time they are different. It is for information surprises to close the gap between them. This issue has lately raised due concern among scholars and practitioners. With professor Elton's own words: " *The use of average realized returns as a proxy for expected returns relies on a belief that information surprises tend to cancel out over the period of a study and realized returns are therefore an unbiased estimate of expected returns. However, I believe that there is ample evidence that this belief is misplaced*" (Elton, 1999).

Therefore, we need to set up foundations in the family of information sets which allow us to properly define not only differential rates but the expectations operator as well. To accomplish such a task we have to expand on rings and algebras of sets.

Remark:

In the framework of discrete modelling with finite algebras and rings of sets, it does not seem essential to make the underlying probability distributions explicit, at least for the environment we are proposing in this paper. Whereas infinite set structures and continuous modelling have been widely undertaken, with strong stress in probability theory, the former approach has been rather neglected.

04.- RINGS OF SETS

Let **R** a non-empty collection of subsets of a set **X** which is usually called a space or universe .

<u>*Definition 1*</u> <u>*Ring of Sets*</u>

R *is called a* ***ring*** *of sets if it satisfies the following properties:*

$$E \in \boldsymbol{R} \;,\; F \in \boldsymbol{R} \quad \Rightarrow \quad E \cup F \in \boldsymbol{R}$$

$$E \in \boldsymbol{R} \;,\; F \in \boldsymbol{R} \quad \Rightarrow \quad E - F \in \boldsymbol{R}$$

Remark:

As a remainder for the reader, the Appendix displays some basic set operations.

- <u>Example 1</u>

  The smallest ring is the class { ∅ } , which contains the empty set only. And the biggest one is the class **P (X)** of all subsets of **X**, because it is a structure closed under unions and differences of sets. Hence, there are at least two rings for any space.

- <u>Example 2</u>

  Let us take two different, non empty sets, E and F, both in **X** . We are going to construct the minimal ring that contains both sets.

The empty set $\varnothing$ is included in such a ring, because it comes out of the difference of E with itself. Both E – F, F – E , must also be included, otherwise we would have not a ring. The same takes place with $E \cup F$. The symmetric difference of E and F is defined as

$$E \,\Delta\, F = (E - F) \cup (F - E)$$

Clearly, it should belong to the ring. Furthermore, the intersection of E and F can be constructed from:

$$E \cap F = (E \cup F) - (E \,\Delta\, F)$$

And that's all, the minimal ring built up from E and F contains eight subsets of **X:**

$$\mathbf{R[}\, E, F \,\mathbf{]} = \{\, \varnothing, E, F, E \cup F, E \cap F, E - F, F - E, E \,\Delta\, F \,\}$$

R[E, F] is to be read "the ring generated by E and F ".

Remark:

With only two sets we get a ring that contains eight components. If E is an information set related to transaction costs, and F is the information set regarding a nominal rate of return for a certain financial asset, then F – E is the information set which provides a rate of return net of transaction costs. When we deal with three subsets or more sets, the ring becomes complex in structure. On this account, see Halmos (1974).

- <u>Example 3</u>

The following class

$$\mathbf{R} = \{\, M \subseteq \mathbf{X} : M \text{ is finite and non empty} \,\}$$

is a ring. To see this, we recall that the union of finite sets are still finite, and the set difference M – N, of members M, N in **R**, is included in M. Therefore, **R** is a ring.

A covering property as shown in example 3 always comes in handy on contexts of applications where we have to cope with several information sets, most of them fuzzily defined or difficult to sort out. The following lemma addresses this issue, while highlighting some important features to put into practice.

<u>Lemma 1</u> (Minimal rings and covering properties)

*Let α be any class of sets in the space **X**. The following statements hold true:*

*i) The class **<** α **>** of all sets in **X** which can be covered by a finite collection of sets in α is a ring.*

*ii) There exists a smallest ring **R [** α **]** which contains α.*

*iii) Every set in **R [** α **]** can be covered by a finite collection of sets in α.*

Proof: i) let us build up the following class of sets in **X**

$$< \alpha > = \{ M \subseteq \mathbf{X} : M \subseteq \bigcup_{1 \leq k \leq N} E_k \; ; \; E_k \in \alpha \}$$

and take any M and any N in **< α >**.Then:

$$M \subseteq \bigcup_{1 \leq k \leq N} E_k \quad ; \quad N \subseteq \bigcup_{1 \leq j \leq M} E_j$$

it follows that

$$M \cup N \subseteq \bigcup_{1 \leq v \leq N+M} E_v \quad \Rightarrow \quad M \cup N \in < \alpha >$$

Furthermore,

$$M - N \subseteq M \subseteq \bigcup_{1 \leq k \leq N} E_k \quad \Rightarrow \quad M - N \in < \alpha >$$

Summing up, **< α >** is a ring.

ii) The set **P(X)** of all subsets of **X** is a ring and the biggest one defined in **X**, then the class of sets α is contained at least by one ring. Let us take as **R [α]** the intersection of all the rings in **X** that contain **α.** It is certainly a ring, and it is contained in any ring which could contains α. Hence, **R [α]** is the smallest of all them; in other words, it is minimal.

iii) Let **< α >** be the class all of sets in **X** which can be covered by a finite union of sets in α. In particular, any set in α belongs to **< α >**. By i) **< α >** is a ring and by ii) it also contains the minimal ring **R [α]** . χ

4.01.- APPLICATION OF LEMMA 1 TO INFORMATION SETS

a) For any family α of information sets, there exists a minimal ring which contains such family.

b) We usually meet a wide array of information sets, ranging from those with very few members, to those families which contains even an infinite number of them. In a wider context of application, a family of well-behaved information sets may available to the researcher but, on the other hand, she might be pursuing another family of information sets not so easy to reach. If the first family can provide a covering feature, the second one would become more manageable eventually.

05.- ALGEBRAS OF SETS

Rings provide with a fairly good structure to deal with information sets. However, in many contexts of analysis and application, a stronger structure could perform even better. In fact, when we work on some set in a ring, we are not granted that its complement belongs to the ring as well. As a ring does not usually contains the space **X,** we proceed to the concept of algebra of sets, which always contains **X.**

Definition 2 *Algebra of Sets*

***A** is called an **algebra** of sets if it satisfies the following properties:*

$$E \in \mathbf{A}\ ,\ \ F \in \mathbf{A} \quad \Rightarrow \quad E \cup F \in \mathbf{A}$$

$$E \in \mathbf{A}\ ,\ \ F \in \mathbf{A} \quad \Rightarrow \quad E^{C} \in \mathbf{A}$$

That is to say, an algebra of sets is an structure closed under unions and complements of sets. Some authors, like Aliprantis (1999) or Halmos (1974), give a stylish translation of an algebra of sets as being any ring containing the space **X**.

- Example 4

The smallest algebra is the class $\{ \emptyset , \mathbf{X} \}$, whereas the biggest one is the class **P (X)** of all subsets of **X**. In fact, **P (X)** is closed under unions and complements, by definition. Hence, there are two algebras in any space at least.

- Example 5

Let us take in **X** two different, non empty sets, E and F. We are going to construct the minimal algebra that contains both sets. Firstly, it must contain **X** because is the complement of the empty set. Also, the complements of E, F and $E \cup F$ must be in such algebra. There are other sets, and to show they should belong to the algebra, we take advantages of very well known set theory properties.

$$E \cap F = (E^{C} \cup F^{C})^{C}$$

$$(E-F) = E \cap F^{C} \quad ; \quad (F-E) = E^{C} \cap F$$

$$E \,\Delta\, F = (E-F) \cup (F-E)$$

If these sets had to belong to the minimal algebra, so would their complements. All in all, there are sixteen sets in the minimal algebra that contains both E and F:

$$\mathbf{A} = \{ \emptyset, \mathbf{X}, E, F, E^{C}, F^{C}, E \cup F, (E \cup F)^{C}, E \cap F, (E \cap F)^{C},$$

$$(E-F), (E-F)^{C}, (F-E), (F-E)^{C}, E \,\Delta\, F, (E \,\Delta\, F)^{C} \}$$

With only two sets we get a ring that contains eight components. As an algebra grants that complements remain within the structure, it comes as no surprise that the eight components of the ring in example 2 are present here, and their complements as well, so as to give sixteen sets in the algebra of sets.

An equivalent set of statements to those of Lemma 1 do not hold true for algebras. Nevertheless, we can follow an easy transition from a ring to an algebra through next lemma, so as to proceed later to another lemma which provides minimal algebras and covering features under somewhat restrictive assumptions.

Lemma 2

If **R** is a ring in the space **X**, then the class of sets

$$\mathbf{A} = \{ \; M \subseteq \mathbf{X} : M \in \mathbf{R} \;\text{ or }\; M^C \in \mathbf{R} \; \}$$

is an algebra.

Proof: Firstly, let us take $M \in \mathbf{A}$. There are two chances:

a) $M = (M^C)^C \in \mathbf{R} \quad \Rightarrow \quad M^C \in \mathbf{A}$

b) $M^C \in \mathbf{R} \quad \Rightarrow \quad M^C \in \mathbf{A}$

Hence, for each set in **A**, its complement also belongs to **A**.

Secondly, suppose M , N are in **A**. There are three distinctive alternatives:

a) $M, N \in \mathbf{R} \quad \Rightarrow \quad M \cup N \in \mathbf{R} \quad \Rightarrow \quad M \cup N \in \mathbf{A}$

b) $M, N^C \in \mathbf{R} \quad \Rightarrow \quad N^C - M = N^C \cap M^C = (M \cup N)^C \in \mathbf{R} \quad \Rightarrow \quad (M \cup N) \in \mathbf{A}$

c) $M^C, N^C \in \mathbf{R} \quad \Rightarrow \quad (M^C \cap N^C)^C = M \cup N \in \mathbf{R} \quad \Rightarrow \quad M \cup N \in \mathbf{A}$

In any case, $(M \cup N) \in \mathbf{A}$

Therefore, **A** qualifies as an algebra. χ

Remarks:

a) In many applications it could be helpful to know that some sets belong to one referential ring R and some others sets, although not belonging to the benchmark ring, grant membership for their complements. Therefore, Lemma 2 shows how both types of sets become members of a definite algebra eventually.

b) Example 3 could not be extended to an algebra whenever X is infinite.

For any family of well-behaved information sets we can generate a ring with covering properties as depicted in Lemma 1. Whereas we can not extend those covering features to any algebra of sets outright, next lemma will show that, with some qualifications, such extension can be reached anyway.

Lemma 3 (minimal algebras and covering features)

*Let α be any class of sets in the space **X**. The following statements hold true:*

i) *There exists a minimal (smallest) algebra $\mathbf{A}[\,\alpha\,]$ which contains the family α.*

ii) *In particular, there exists a smallest algebra $\mathbf{A}[\,\mathbf{R}[\alpha\,]\,]$ which contains the ring generated by the family α.*

iii) $\mathbf{A}[\,\mathbf{R}[\alpha]\,] = \{\, M \subseteq \mathbf{X} \;:\; M \in \mathbf{R}[\alpha] \;\text{ or }\; M^C \in \mathbf{R}[\alpha] \,\}$

iv) *If **A** is an algebra, then it is also a ring.*

v) *If **R** is a ring which contains **X**, then it is an algebra.*

vi) *Let α be a family of subsets of **X** which contains **X** itself. Then $\mathbf{R}[\,\alpha\,]$ is an algebra with covering features as in Lemma 1.*

Proof:

i) Certainly, the family α is contained in **P(X)**, the largest algebra as was shown in example 4 above. Let $\mathbf{A}[\,\alpha\,]$ the intersection of all algebras which contain the family α. For every E, F taken in $\mathbf{A}[\,\alpha\,]$, their union and complements belong to it because that happens to any algebra involved. Minimality follows outright because any algebra contained in $\mathbf{A}[\,\alpha\,]$ it would also contain the family α and hence $\mathbf{A}[\,\alpha\,]$.

ii) By choosing the family $\mathbf{R}[\alpha\,]$, the statement follows from i).

iii) By Lemma 2, the family of sets

$$\{\, M \subseteq \mathbf{X} \;:\; M \in \mathbf{R}[\alpha] \;\text{ or }\; M^C \in \mathbf{R}[\alpha] \,\}$$

is an algebra. By i) and ii) above, it contains $\mathbf{A}[\,\mathbf{R}[\alpha\,]\,]$. Next, let us take any algebra **A** which contains $\mathbf{A}[\,\mathbf{R}[\alpha\,]\,]$. It holds that:

$$M \subseteq \mathbf{X} \text{ and } M \in \mathbf{R}[\alpha] \;\Rightarrow\; M \in \mathbf{A}$$

$$M \subseteq \mathbf{X} \text{ and } M^C \in \mathbf{R}[\alpha] \;\Rightarrow\; M^C \in \mathbf{A}$$

Hence, the family is included in **A**. That is to say:

$$\{\, M \subseteq \mathbf{X} \;:\; M \in \mathbf{R}[\alpha] \;\text{ or }\; M^C \in \mathbf{R}[\alpha] \,\} \subseteq \mathbf{A}$$

Therefore, this is the minimal algebra which contains $\mathbf{R}[\,\alpha\,]$ and this means it is $\mathbf{A}[\,\mathbf{R}[\,\alpha\,]\,]$.

iv) Being an algebra, **A** is closed under unions. Besides, for every E, F in **A,**

$$E \in \mathbf{A}, \quad F \in \mathbf{A} \;\Rightarrow\; E - F = E \cap F^{C} \in \mathbf{A}.$$ Hence, **A** is also a ring.

v) For every set E in the ring **R**, it follows

$$E^{C} = \mathbf{X} - E \in \mathbf{R}$$

Hence, such a ring becomes an algebra.

vi) Let α be a family of subsets of **X** which contains **X** itself. By Lemma 1, $\mathbf{R}[\,\alpha\,]$ is a ring with covering features and it contains **X**. Besides, by v) it is an algebra. . χ

5.01.- APPLICATION OF LEMMA 3 TO INFORMATION SETS

a) For any family of information sets there exists a minimal algebra which contains α.

b) Furthermore, in many contexts of application, when we work on a family of feasible information sets, it does not seem implausible to assume that X itself actually belongs to the family. In that way, by means of Lemma 3 we can handle minimal algebras with covering features.

06.- DIFFERENTIAL RATES OF RETURN

Getting access to any information set could lead to finding out different sources which could explain the rate of change $\mathbf{r}(\,t\,,\,T\,,\,\Omega_t\,)$ defined on the basic information set Ω_t and measured as in [02]. What, for instance, if we knew that there is a subset $\Omega^1{}_t$ of Ω_t, namely,

$$\Omega^{1}{}_{t} \subseteq \Omega_{t}$$

which is so influential that could explain two thirds of the value of $\mathbf{r}(\,.\,)$ at least? Let address two examples for this to happen. More applications, mainly in the financial markets context, can be found in Apreda (2000b).

- Example 7

  Assuming that we are about to value a financial asset and we know that the expected inflation rate will add one percentual point to the nominal rate of return for the period under analysis, here $\Omega^1{}_t$ accounts for the information set that deals with inflation data. It is for the remaining of the nominal $\mathbf{r}(\,.\,)$ to track down transaction costs and the effective yield for the investor, exclusive of inflation, and the information would be supplied by the difference between the basic set and $\Omega^1{}_t$.

- <u>Example 8</u>

  Let us suppose that we try to isolate variable costs attached to some measure of change, for instance, profit rate per unit of a gadget to be sold. All the knowledge we need to calculate such a cost component is gathered into $\mathbf{\Omega^1{}_t}$. It is for the remaining of the variable under study to track down fixed costs, semi-fixed costs, profits, and any other relevant item to the needs of the analyst. Again, the information would be drawn from the difference

$$\mathbf{\Omega_t - \Omega^1{}_t}$$

It is within this setting, as both Examples 7 and 8 have shown, that it becomes advisable to split down the original rate of return into two components:

$$\mathbf{1 + r(t, T, \Omega_t) = [\, 1 + s^1(t, T, \Omega^1{}_t)\,] \,.\, [\, 1 + g^1(.)\,]}$$

where

$$\mathbf{s^1(t, T, \Omega^1{}_t)}$$

stands for the rate of change conditional to the information set $\mathbf{\Omega^1{}_t}$. Furthermore, $\mathbf{g^1(\,.\,)}$ comes up by solving the equation and stands for whatever remains of $\mathbf{r(\,.\,)}$ after taking into account $\mathbf{s^1(\,.\,)}$. So, this complementary rate $\mathbf{g^1(\,.\,)}$ fills the gap between both rates and it is conditional upon the information set

$$\mathbf{\Omega_t - \Omega^1{}_t}$$

That means:

$$\mathbf{g^1(.) = g^1(t, T, \Omega_t - \Omega^1{}_t)}$$

Hence, we can now restate the former relationship between these three rates of change:

[04]

$$\mathbf{1 + r(t, T, \Omega_t) = [\, 1 + s^1(t, T, \Omega^1{}_t)\,] \,.\, [\, 1 + g^1(t, T, \Omega_t - \Omega^1{}_t)\,]}$$

The discussion above has paved the way to the following definition. Although this definition keeps a very simple format lying on a general algebra of sets, we will proceed to uncover the underlying structure of such an algebra in section 06.01.

<u>Definition 3</u>: *<u>Simple Differential Rates on Residual Information Sets</u>*

Let $\boldsymbol{A}$ *be an algebra of sets in* $\boldsymbol{X}$*. Given the rates of return* $\boldsymbol{r(t, T, \Omega_t)}$ *and* $\boldsymbol{s^1(t, T, \Omega^1{}_t)}$ *such that* $\boldsymbol{\Omega^1{}_t \subseteq \Omega_t}$ *, it is said that* $\boldsymbol{g^1(\,.\,)}$ *is a simple differential rate to both* $\boldsymbol{r(\,.\,)}$ *and* $\boldsymbol{s^1(\,.\,)}$ *if and only if*

$$\boldsymbol{g^1(.) = g^1(t, T, \Omega_t - \Omega^1{}_t)}$$

and it fulfils:

$$1 + r(t, T, \Omega_t) = [1 + s^1(t, T, \Omega^1{}_t)] . [1 + g^1(t, T, \Omega_t - \Omega^1{}_t)]$$

Furthermore, the set $\Omega_t - \Omega^1{}_t$ *will be called a residual information set.*

Remarks:

i) We see that **$g^1(.)$** is the remainder of the leading rate **$r(.)$**, given **$s^1(.)$**.

ii) Residual sets come into light only because we have an underlying algebra which provides closure for differences. Although rings also allow such closure, oncoming developments will require complements to be used and, therefore, algebras become more flexible structures.

iii) When plugging numbers in definition 3, it may hold true that

$$[1 + s^1(t,T,\Omega^1{}_t)].[1 + g^1(t,T,\Omega^g{}_t)] = [1 + s^1(t,T,\Omega^1{}_t)].[1 + h^1(t,T,\Omega^h{}_t)]$$

which implies

$$g^1(t,T,\Omega^g{}_t) = h^1(t,T,\Omega^h{}_t)$$

but this fact doesn't convey that, as functions, $g^1(.) = h^1(.)$, because they can be defined on different information sets.

iv) It is theformer remark that leads to a good reason for working with a cartesian product

$$R^1 \times R^1 \times A$$

as it is usual, for instance, in mathematical treatments on stochastic processes. (Oksendal, 1985)

What if we now tried to deal with **$g^1(.)$** the same way we did with **$r(.)$**, by resorting to the simple differential rates definition? That is to say, what if knew that we can pick out another rate **$s^2(.)$** conditional upon the information set $\Omega^2{}_t$ that fulfills:

$$\Omega^2{}_t \subseteq \Omega_t \quad , \quad \Omega^2{}_t \cap \Omega^1{}_t = \varnothing$$

In that case we should ask for the remainder of **$g^1(.)$**, and try to solve:

$$1 + g^1(t,T,\Omega_t - \Omega^1{}_t) = [1 + s^2(t,T,\Omega^2{}_t)] . [1 + g^2(.)]$$

Following this way, we may obtain a finite vector of rates of change

$$< s^1(.), s^2(.), \ldots\ldots, s^N(.) >$$

all of them stemming from the primary rate **$r(.)$**, and also a differential rate **$g^N(.)$** performing as a remainder, so as the following relationship must hold true by iteration:

$$1 + r(t,T, \Omega_t) = \{ \Pi_{1 \leq k \leq N} [1 + s^k(t,T, \Omega^k{}_t)] \} . [1 + g^N(.)]$$

Now, a problem certainly arises with this expression, because we have still said nothing on the underlying information sets of the components in the following finite vector

$$< g^1(.) , g^2(.) , ... , g^N(.) >$$

To handle this difficulty, we need to enlarge upon Definition 2. Such a task will be accomplished through two successive approaches. The first will lead to a narrower meaning of differential rate, whereas the second one will address an even broader but more realistic meaning, .

However, before expanding on what is understood by narrow and broad differential rates, we have to agree on the actual algebra of sets to be used as from now.

06.01.- CHOOSING THE RELEVANT ALGEBRA OF INFORMATION SETS

In practice, we choose a primary information set Ω_t in the space **X**, and we are concerned with a certain family of contextual sets in **X**, which might amount to economic variables or transaction costs structures. It is the purpose of this section to show how a family of relevant sets may be spanned into a suitable minimal algebra. An example will shed light about this sort of sets.

While dealing with financial assets rates of return, we should be intent on making the analysis inclusive of the transaction costs structure. At least, five contextual subsets of **X** seems to be useful (this approach has been developed in Apreda, 2000a and 2000b) :

- Intermediation costs : INT
- Microstructure costs : MICR
- Financial costs on transactions : FIN
- Information costs : INF
- Taxes : TAX

We define a family of contextual sets to Ω_t :

$$\alpha = \{ E , F , G , H , J \}, \qquad \text{where}$$

$$E = \Omega_t \cap INT ; F = \Omega_t \cap MICR ; G = \Omega_t \cap FIN ; H = \Omega_t \cap INF ; J = \Omega_t \cap TAX$$

The minimal algebra which is spanned by these sets and Ω_t. will meet our purposes. We denote this algebra as $\mathbf{A}[\Omega_t , \alpha]$. That is to say:

$$\mathbf{A}[\Omega_t , \alpha] = \mathbf{A}[E, F, G, H, J, \Omega_t]$$

If we want to go beyond transaction costs, the class could be embedded into a larger one, with more factors of analysis. For instance, inflation, rate of exchange, and so on. Making for a definition, we get:

<u>*Definition 4:*</u> <u>*The relevant algebra for information sets*</u>

*Given $\Omega_t \subseteq X$, and a finite family of sets in **X** ,*

$$\alpha = \{ E_1, E_2, E_3, \ldots, E_N \}$$

the algebra $A[\Omega_t, \alpha]$ *spanned by the family*

$$\{ E_1 \cap \Omega_t, E_2 \cap \Omega_t, E_3 \cap \Omega_t, \ldots, E_N \cap \Omega_t, \Omega_t \}$$

it will be called the relevant algebra to the information set Ω_t *, subject to the contextual famiy* α.

06.02.- NARROW DIFFERENTIAL RATES OF RETURN

Once we get a single differential rate, to what extent can we add other differential rates which could improve our knowledge of the primary rate, $r(t, T, \Omega_t)$? It is for successive differential rates stemming from the residual information set at each stage, to claim for an iterative algorithm. That is to say, we have to deal with a sort of splitting-down decision problem, which is addressed by next lemma.

Lemma 4

Let $A[\Omega_t, \alpha]$ *be any relevant algebra for the information set* Ω_t *. Then, the following statements hold true:*

i) For every pair of rates of return $r(t,T,\Omega_t)$ *and* $s^1(t,T,\Omega^1{}_t)$ *such that*

$$\Omega^1{}_t \subseteq \Omega_t \in A[\Omega_t, \alpha]$$

there is a $g^1(.)$ *which is a differential rate to both* $r(.)$ *and* $s^1(.)$

ii) For every finite vector of rates of return,

$$< s^1(t,T,\Omega^1{}_t), s^2(t,T,\Omega^2{}_t), \ldots, s^k(t,T,\Omega^k{}_t) > \text{, such that}$$

$$\Omega^j{}_t \in A[\Omega_t, \alpha]; \quad \Omega^j{}_t \cap \Omega^i{}_t = \varnothing \quad (j, i: 1,2, \ldots, k)$$

there is a $g^k(.)$ *which is differential rate to the preceding rates of return.*

Proof:

i) We take

$$g^1(.) = g^1(t,T,\Omega_t - \Omega^1{}_t)$$

defined by:

$$[1 + r(t,T,\Omega_t)] \,/\, [1 + s^1(t,T,\Omega^1{}_t)] = [1 + g^1(t,T,\Omega_t - \Omega^1{}_t)]$$

and $\mathbf{g^1(\,.\,)}$ turns out to be a well defined function because its set of information belongs to the underlying algebra of information sets.

ii) The proof is inductive . It has been shown in i) that the statement holds true when k = 1 in the rate $\mathbf{g^1(\,.\,)}$

Let us suppose that the statement holds true for k = N, that is to say:

$$\mathbf{1 + r(t, T, \Omega_t) =}$$

$$\mathbf{\{ \Pi_{1\le k \le N} [1 + s^k(t, T, \Omega^k{}_t)] \} . [1 + g^N(t, T, \Omega_t - \cup_{1\le k \le N} \Omega^k{}_t)]}$$

and a new information set $\mathbf{\Omega^{N+1}{}_t}$ that fulfils

$$\mathbf{\Omega^{N+1}{}_t \in A[\Omega_t , \alpha] ; \quad \Omega^j{}_t \cap \Omega^i{}_t = \varnothing} \qquad (j, i : 1,2, \dots , N+1)$$

is predicated such that the rate $\mathbf{g^N(\,.\,)}$ allows for

$$\mathbf{[1 + g^N(t, T, \Omega_t - \cup_{1\le k \le N} \Omega^k{}_t)] =}$$

$$\mathbf{[1 + s^{N+1}(t, T, \Omega^{N+1}{}_t)] . [1 + g^{N+1}(t, T, \Omega_t - \cup_{1\le k \le N+1} \Omega^k{}_t)]}$$

It follows that

$$\mathbf{1 + r(t, T, \Omega_t) =}$$

$$\mathbf{\{ \Pi_{1\le k \le N+1} [1 + s^k(t,T,\Omega^k{}_t)] \} . [1 + g^{N+1}(t,T,\Omega_t - \cup_{1\le k \le N+1} \Omega^k{}_t)]}$$

By complete induction, ii) holds true for every k. χ

It is worthy of remark that the iterative algorithm in the proof of Lemma 4 may stop at some stage, let us say k = N. This could happen whenever the N information sets explain $\mathbf{r(\,.\,)}$ exhaustively. In such a case:

$$\mathbf{g^N(t, T, \Omega_t - \cup_{1\le k \le N} \Omega^k{}_t) = 0}$$

Last of all, Lemma 4 clears up the meaning of a narrow differential rate, leading to the following definition.

Definition 5: *Narrow Differential Rates on Residual Information Sets*

Let $\boldsymbol{A[\Omega_t , \alpha]}$ *be certain relevant algebra for the primary information set* $\boldsymbol{\Omega_t}$ *. Given a rate of return* $\boldsymbol{r(t, T, \Omega_t)}$ *defined on* $\boldsymbol{\Omega_t}$*, and a finite vector of rates of return,*

$$\boldsymbol{< s^1(t,T, \Omega^1{}_t), s^2(t,T, \Omega^2{}_t), \dots, s^N(t,T, \Omega^N{}_t) >}$$

such that $\Omega^{k}{}_{t} \subseteq \Omega_{t}$ *and* $\Omega^{j}{}_{t} \cap \Omega^{k}{}_{t} = \varnothing$ *for j, k : 1, 2, ... , N*

it is said that $\mathbf{g}^{k}(\,.\,)$ *is a narrow differential rate whenever its information set becomes*

[05]

$$\Omega_{t} - \cup_{1 \leq k \leq N}\ \Omega^{k}{}_{t}$$

06.03.- BROAD DIFFERENTIAL RATES OF RETURN

We are going to deal with an alternative approach to the one previously set forth in Lemma 4, It will lie on differently shaped residual information sets. This approach leads to the notion of a "broad differential rate", and it will be expanded later on in Lemma 5. Before proceeding to the lemma, however, some preliminary work will be set up as a constructive pathway to such main outcome.

Stage 1: Firstly, we index the algebra components at valuation date " t ".

$$\mathbf{A}[\,\Omega_{t}\,,\alpha\,] = \{\ \Omega^{p}{}_{t} : p \in \mathbf{I}\ \}$$

where **I** is a finite interval of positive integers.

Stage 2: Next, we pick up a member $\Omega^{p(1)}{}_{t} \subseteq \Omega_{t}$ in the algebra $\mathbf{A}[\,\Omega_{t}\,,\alpha\,]$, which is relevant to account for the rate of return

$$\mathbf{s}^{1}(\,t\,,T\,,\ \Omega^{p(1)}{}_{t})$$

Now, we want to exclude from $\Omega^{p(1)}{}_{t}$ any point which could be shared with another member of the algebra $\mathbf{A}[\,\Omega_{t}\,,\alpha\,]$. This lead to the set

[06]

$$\Omega^{p(1)}{}_{t} - \cup\{\ \Omega^{p}{}_{t} : p \in I\,;\, p \neq p(1)\ \}$$

Stage 3: It is a well known property of any arbitrary collection of sets defined in certain universe that

[07]

$$\mathbf{B} - [\,\mathbf{A}_{1} \cup \mathbf{A}_{2} \cup \mathbf{A}_{3} \cup \ldots\ldots \cup \mathbf{A}_{N}\,] =$$

$$= (\mathbf{B} - \mathbf{A}_{1}) \cap (\mathbf{B} - \mathbf{A}_{2}) \cap (\mathbf{B} - \mathbf{A}_{3}) \cap \ldots\ldots\ldots \cap (\mathbf{B} - \mathbf{A}_{N})$$

$$= \cap\ \{\,(\mathbf{B} - \mathbf{A}_{k}) : k = 1, 2, 3, \ldots. , N\,\}$$

By applying [07] to [06] we get:

[08]

$$\Omega^{p(1)}{}_{t} - \cup\{\ \Omega^{p}{}_{t} : p \in I\,;\, p \neq p(1)\ \} =$$

$$= \cap\{\ (\Omega^{p(1)}{}_{t} - \Omega^{p}{}_{t}) : p \in I\,;\, p \neq p(1)\ \}$$

Stage 4 : In the equation

$$1 + r(t, T, \Omega_t) = [\, 1 + s^1(t, T, \Omega^{p(1)}_t) \,] \,.\, [\, 1 + g^1(.) \,]$$

and taking advantage of [08], we will choose as the underlying residual information set for $g^1(.)$

[09]

$$\Omega_t - \cap\{ (\Omega^{p(1)}_t - \Omega^p_t) : p \in I ; p \neq p(1) \}$$

This set brings back all the elements in $\Omega^{p(1)}_t$ which could be of interest for any other information set relevant to following differential rates. That is to say, $g^1(.)$ is defined on all the points of Ω_t including those points in the sets overlapping with $\Omega^{p(1)}_t$.

Stage 5 : We labour [09] by means of properties that hold for arbitrary collections of sets, namely:

$$\Omega_t - \cap\{ (\Omega^{p(1)}_t - \Omega^p_t) : p \in I ; p \neq p(1) \} =$$

$$= \Omega_t \cap < \cap\{ (\Omega^{p(1)}_t - \Omega^p_t) : p \in I ; p \neq p(1) \} >^C$$

$$= \Omega_t \cap < \cup\{ (\Omega^{p(1)}_t - \Omega^p_t)^C : p \in I ; p \neq p(1) \} >$$

$$= \Omega_t \cap < \cup\{ (\Omega^{p(1)}_t \cap (\Omega^p_t)^C)^C : p \in I ; p \neq p(1) \} >$$

$$= \Omega_t \cap < \cup\{ ((\Omega^{p(1)}_t)^C \cup \Omega^p_t) : p \in I ; p \neq p(1) \} >$$

$$= < \Omega_t \cap (\Omega^{p(1)}_t)^C > \cup < \cup\{ (\Omega_t \cap \Omega^p_t) : p \in I ; p \neq p(1) \} >$$

$$= < \Omega_t - \Omega^{p(1)}_t > \cup < \cup\{ (\Omega_t \cap \Omega^p_t) : p \in I ; p \neq p(1) \} >$$

Summing up, the residual information set for $g^1(.)$ will be:

[10]

$$\Omega_t - \cap\{ (\Omega^{p(1)}_t - \Omega^p_t) : p \in I ; p \neq p(1) \} =$$

$$= < \Omega_t - \Omega^{p(1)}_t > \cup < \cup\{ (\Omega_t \cap \Omega^p_t) : p \in I ; p \neq p(1) \} >$$

This relationship is worthy of remark:

(a) The first part of it, namely

$$< \Omega_t - \Omega^{p(1)}_t >$$

is the same as in [05], that is to say, the residual set we set up in the narrow approach to differential rates.

(b) The second part of it, namely

$$< \cup\{ (\Omega_t \cap \Omega^p{}_t) : p \in I ; p \neq p(1) \} >$$

conveys the sharing of information with all overlapping sets, as stand-by remainders to be used by other differential rates. As we see, residual sets are not so restrictive here as they were in the narrow approach to differential rates.

Stage 6 : At this stage, we are going to pick another member of $A[\Omega_t , \alpha]$,

$$\Omega^{p(2)}{}_t \subseteq \Omega_t ,$$

which is relevant to account for the rate of return

$$s^2 (t , T , \Omega^{p(2)}{}_t)$$

Up to this point, we would like to isolate from $\Omega^{p(2)}{}_t$ any point which could be shared with another member of $A[\Omega_t , \alpha]$ eventually. As we are in the second rate of return, another $\Omega^{p(1)}{}_t$ has been previously taken into account. So, we want to isolate from

$$\Omega^{p(1)}{}_t \cup \Omega^{p(2)}{}_t$$

any point which could be shared with another member of $A[\Omega_t , \alpha]$. This leads to the set

$$(\Omega^{p(1)}{}_t \cup \Omega^{p(2)}{}_t) - \cup\{ \Omega^p{}_t : p \in I ; p \neq p(1) , p(2) \}$$

By using [07] this set can also be translated as

$$(\Omega^{p(1)}{}_t \cup \Omega^{p(2)}{}_t) - \cup\{ \Omega^p{}_t : p \in I ; p \neq p(1) , p(2) \} =$$

$$= \cap\{ [(\Omega^{p(1)}{}_t \cup \Omega^{p(2)}{}_t) - \Omega^p{}_t] : p \in I ; p \neq p(1) , p(2) \}$$

Stage 7 : Next, we have to solve for the new differential rate, as we did in stage 4:

$$1 + g^1 (.) = [1 + s^2 (t , T , \Omega^{p(2)}{}_t)] . [1 + g^2 (.)]$$

and we proceed to choose the residual information set of $g^2 (.)$

[12]

$$\Omega_t - \cap\{ [(\Omega^{p(1)}{}_t \cup \Omega^{p(2)}{}_t) - \Omega^p{}_t] : p \in I ; p \neq p(1) , p(2) \}$$

This set brings back all the elements in $(\Omega^{p(1)}{}_t \cup \Omega^{p(2)}{}_t)$ which could be of interest for any other information set relevant to the following differential rates. That is to say, $g^2 (.)$ is defined on all the points of Ω_t including those points in the sets overlapping with

$$(\Omega^{p(1)}{}_t \cup \Omega^{p(2)}{}_t).$$

we get a working expression of [12] , through the same steps we did in stage 5:

[13]

$$\mathbf{\Omega_t - \cap\{ [(\Omega^{p(1)}_t \cup \Omega^{p(2)}_t) - \Omega^p_t] :} p \in I ; p \neq p(1), p(2) \mathbf{\} =}$$

$$\mathbf{= < \Omega_t - (\Omega^{p(1)}_t \cup \Omega^{p(2)}_t) > \cup < \cup\{ (\Omega_t \cap \Omega^p_t) :} p \in I ; p \neq p(1), p(2) \mathbf{\} >}$$

It should not come as a surprise that the first part in the last expresion, namely:

$$\mathbf{< \Omega_t - (\Omega^{p(1)}_t \cup \Omega^{p(2)}_t) >}$$

is the same as in [5], that is to say the same residual set used in the narrow approach to differential rates. On the other hand, the second part in the last expression amounts to

$$\mathbf{< \cup\{ (\Omega_t \cap \Omega^p_t) :} p \in I ; p \neq p(1), p(2) \mathbf{\} >}$$

it conveys information sharing with overlapping sets, as stand-by remainders to be used by other differential rates.

Taking advantage of this rather lengthy discussion, next lemma can be easily set forth.

<u>Lemma 5</u>

Let $\boldsymbol{A[\Omega_t , \alpha]}$ be the relevant algebra to the information set $\boldsymbol{\Omega_t}$, indexed by a finite interval $\boldsymbol{I}$ of positive integers,

$$\boldsymbol{A[\Omega_t , \alpha] = \{ \Omega^p_t : p \in I \}}$$

Then, the following statements hold true:

i) For every pair of rates of return $\boldsymbol{r(t, T, \Omega_t)}$ and $\boldsymbol{s^1(t, T, \Omega^1_t)}$ such that

$$\boldsymbol{\Omega^1_t \subseteq \Omega_t \in A[\Omega_t , \alpha]}$$

there is one $\boldsymbol{g^1(\,.\,)}$ which is a differential rate to both $\boldsymbol{r(\,.\,)}$ and $\boldsymbol{s^1(\,.\,)}$

ii) For every finite vector of rates of return,

$$\boldsymbol{< s^1(t, T, \Omega^1_t), s^2(t, T, \Omega^2_t), \ldots, s^k(t, T, \Omega^k_t) >}$$

$$\textit{such that} \quad \boldsymbol{\Omega^j_t \in A[\Omega_t , \alpha]} \qquad (j : 1, 2, \ldots, k)$$

there is one $\boldsymbol{g^k(\,.\,)}$ which is differential rate to the preceding rates of return.

Proof:

We are going to proceed by induction on k. If k = 1 , then we use stages 1 to 5 from the previous discussion. In fact, from [11] we have:

$$\Omega_t - \cap\{ (\Omega^{p(1)}_t - \Omega^p_t) : p \in I \,;\, p \neq p(1) \} =$$

$$= < \Omega_t - \Omega^{p(1)}_t > \cup < \cup\{ (\Omega_t \cap \Omega^p_t) : p \in I \,;\, p \neq p(1) \} >$$

Next, let us suppose that the statement ii) holds true for any k. We need to prove that it also holds true for (k + 1). If we take advantage of stages 6 to 8 from the previous discussion, the residual information set of $g^k(\,.\,)$ translates into this format:

$$\Omega_t - \cap\{ [\cup\{ \Omega^{p(j)}_t : j = 1, 2, \ldots , k\} - \Omega^p_t] : p \in I \,;\, p \neq p(1) , p(2), \ldots p(k) \} =$$

$$= < \Omega_t - [\cup\{\Omega^{p(j)}_t : j = 1,2, \ldots ,k\}> \cup < \cup\{(\Omega_t \cap \Omega^p_t) : p \in I \,;\, p \neq p(1),p(2),\ldots ,p(k)\}>$$

Last of all, to deal with (k+1) we have to take into account what we did at stage 6 as the way to add another rate in the k-dimension vector and the outcome follows from stage 8 outright, by complete induction on " k " . χ

Remarkably, Lemma 5 enables us to make clear the meaning of a broad differential rate, by means of the following definition.

Definition 6: *Broad Differential Rates on Residual Information Sets*

Let $A[\,\Omega_t\,,\,\alpha\,]$ *be the relevant algebra for the information set* Ω_t*, indexed by a finite interval* I *of positive integers,*

$$A[\Omega_t\,,\alpha] = \{ \Omega^p_t : p \in I \}$$

Given a finite vector of rates of return,

$$< s^1(t, T, \Omega^{p(1)}_t) , s^2(t ,T, \Omega^{p(2)}_t) , \ldots, s^N(t ,T, \Omega^{p(N)}_t) >$$

$$\textit{such that } \Omega^{p(k)}_t \subseteq \Omega_t \textit{ for } k: 1, 2, \ldots , N$$

it is said that $g^N(\,.\,)$ *is a broad differential rate whenever its residual information set becomes*

[14]

$$< \Omega_t - \cup\{ \Omega^{p(j)}_t : j = 1,2, \ldots ,N\}> \cup < \cup\{ (\Omega_t \cap \Omega^p_t) : p \in I \,;\, p \neq p(1),p(2),\ldots ,p(N)\}>$$

07.- CONCLUSIONS

This paper provides a discrete modelling framework, lying on finite rings and algebras of sets, to deal with information sets. In this way, primary information sets linked with a rate of return can be broken down into strings of residual information sets. For instance, this is certainly the case with transaction costs structures.

The main contributions can be found in section 06, where two lemmas are proved. The first one makes possible the use of differential rates of return with restrictive assumptions on their information sets, as most customary applications seem to have need of. The second lemma attempts to get a broader outcome, so as to cope with differential rates lying on realistic residual information sets. Lastly, both lemmas pave the way to definitions of narrow and broad differential rates as residual information sets.

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APPENDIX

Given a space or universe **X**, non empty, a subset E of **X** will be defined as a set such that:

$$(\forall x): \quad x \in E \quad \Rightarrow \quad x \in \mathbf{X}$$

The set built up with all the subsets of **X** will be denoted as **P(X)** and it comes defined as:

$$\mathbf{P(X)} = \{ A : A \subseteq \mathbf{X} \}$$

When dealing with sets of sets, it is rather preferred to speak about families of sets. Then, **P(X)** is a family of sets. Let us take two arbitrary subsets A and B in **X**. The following operations between them build up new sets.

Union of two sets:

$$\mathbf{A} \cup \mathbf{B} = \{ x : x \in A \text{ or } x \in B \}$$

Intersection of two sets:

$$\mathbf{A} \cap \mathbf{B} = \{ x : x \in A \text{ and } x \in B \}$$

Difference of two sets:

$$\mathbf{A} - \mathbf{B} = \{ x : x \in A \text{ and } x \notin B \}$$

Symmetric difference of two sets:

$$\mathbf{A} \,\Delta\, \mathbf{B} = \{ x : x \in A - B \text{ or } x \in B - A \}$$

Complement of a set:

$$\mathbf{A}^{\mathbf{C}} = \{ x : x \notin A \}$$

Family of sets:

A family, or class, of sets in the space X, indexed by the index set I, is a function

$$\mathbf{F} : \mathbf{I} \rightarrow \mathbf{X}$$

Usually, the family is written:

$$\{ \mathbf{A}_\lambda : \lambda \in \mathbf{I} \}$$

The members in the family are the sets $\mathbf{A}_\lambda = \mathbf{F}(\lambda)$

- For an excellent introduction to logic, from an algebraic perspective, is highly remarkable the latest book by Halmos-Givant (1998).

- Very useful books on advanced mathematics, both suitable for economists and financial economics with good mathematics background, are Aliprantis and Border (1999), and Berge's book on topological spaces. (Berge, 1997).