DIFFERENTIAL RATES AND TRANSACTION COSTS

A Toolkit for Practitioners, Accountants and Financial Economists

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Abstract

It is our main concern in this paper to make for the following stages: a) Firstly, stock and flow differential rates will be introduced. Secondly, reverse differential rates are expanded on. Then, the transaction costs function will be featured. As long as we proceed with these issues, fully solved examples are supplied. b) Next, we handle direct applications to financial markets securities. Namely, time deposits, zero coupon bonds, foreign currencies. In each case, not only intuitive acquaintance with the subject is given, but foundations and examples as well. The foundations are conveyed by six lemmas which also give computation guidelines to be used in practice. Among the main conclusions we can draw from this research, two of them deserve due attention: First, microstructure, trading and information should be given serious regard because they could have the last word when we attempt to pick out the real winners among market securities. Second, differential and reverse differential rates have a say whenever we want to know about what remains of financial rates of returns after transaction costs.

JEL : G3, M4

Key words: Transaction costs, differential rates, financial assets returns.

01.- INTRODUCTION

It was shown in a previous paper (Apreda 2000-a) that a suitable framework to deal with transaction costs is provided by the following basic sources of costs:

- Intermediation
- Microstructure
- Information
- Taxes
- Financial costs associated with transactions

Taking advantage of that framework, rates of return of financial assets can be netted from transaction costs. It is our main concern in this paper to make for the following stages:

a) Firstly, the stock and flow differential rates will be introduced. Secondly, the reverse differential rates are expanded on with certain detail. Then, the transaction costs function will be featured. As long as we proceed with these issues, fully solved examples are supplied.

b) Next, we handle direct applications to financial markets securities. Namely, time deposits, zero coupon bonds, foreign currencies. In each case, not only intuitive acquaintance with the subject is given, but foundations and examples as well. The foundations are conveyed by six lemmas which also give computation guidelines to be used in practice.

This paper should be regarded as a complement to the former one, "A Transaction Costs Approach to Financial Assets Rates of Return" (Apreda, Working Paper 161, Universidad del Cema), although it has been written so as to allow for a self contained reading. It is mainly addressed to financial economists, accountants and practitioners. Furthermore, a previous draft has come in handy in some Master courses taught by the author at Universidad del Cema in 1998 and 1999.

02.- DIFFERENTIAL RATES

Let us suppose that in forecasting a rate of interest for a financial asset at valuation date **" t "**, which is to be held along the horizon **[t ; T]**, we get eventually:

$$E\,[\,r(\,t\,,T\,,\Omega_t\,)\,]$$

Our assessment hangs on the available information at " **t** ", which we denote as the information set Ω_t.

Ex~post, however, the actual rate of interest at valuation date **" T "** comes up as

$$r(\,t\,,T\,,\Omega_T\,)$$

where the latter information set also includes Ω_t, which means that information has improved along the horizon and past information has not been not destroyed.

If we wished to measure the gap, $\mathbf{g(t, T)}$, arising between both rates, we should solve:

[01]

$$< 1 + E[\, r(t, T, \Omega_t)\,] > . < 1 + g(t, T) > \; = \; < 1 + r(t, T, \Omega_T) >$$

This gap rate is a good example of what is meant by a differential rate. However, there might be other contexts where this type of rate comes in handy.

What [01] really conveys is that the gap of information between Ω_t and Ω_T can be quantified as a measure for the error made at assuming the rate forecast. This is a customary procedure for any budgetary task. The differential rate $\mathbf{g(t, T)}$ not only closes the gap between the ex~ante and the ex~post values, but provides with a performance measure as well.

But let us take now another perspective where, at the valuation date **" t "** a specific cost item is assessed by means of the rate

$$r(t, T, \Omega^1{}_t)$$

and, at the same valuation date, we also have the assessment of a " global cost " item, inclusive of the former one,

$$r(t, T, \Omega^2{}_t)$$

It follows that the specific cost lies in a smaller information set than the global cost, that is to say:

$$\Omega^1{}_t \subseteq \Omega^2{}_t$$

There will be a gap between both rates, which comes out of what is not accounted for the smaller information set. Such a gap may be measured by a differential rate $\mathbf{g(t, T)}$, which is defined as

[02]

$$< 1 + r(t, T, \Omega^1{}_t) > . < 1 + g(t, T) > \; = \; < 1 + r(t, T, \Omega^2{}_t) >$$

It is worthy of remark how different the contex of this example is from the former one. In [01] we have two valuation dates with their own specific information sets. In [02] we deal with only one valuation date and two specific information sets. In other words, the first example shows a "flow differential rate", and the second example highlights a "stock differential rate".

These examples have paved the way for formal definitions of two broad classes of differential rates. As from now, whenever we wish to highlight any rate without stressing its inner structure, we will just write $\mathbf{r(\,.\,)}$ or $\mathbf{g(\,.\,)}$.

Remarks on notation:

- The gap includes not only the starting and final valuation dates, but an explicit remark on the shift from one information set to the other.
- $\Omega^1{}_t \to \Omega^2{}_t$ means that the gap fills in what is not accounted for by $\Omega^1{}_t$ but it is accounted for by $\Omega^2{}_t$.
- $\Omega^1{}_t \to \Omega^2{}_T$ means that the gap fills in what is not accounted for by $\Omega^1{}_t$ at valuation date **" t "**, but it is accounted for by $\Omega^2{}_T$, at valuation date **" T ".**

Definition 1 <u>*Stock Differential Rate*</u>

Given r(t, T, $\Omega^1{}_t$) and r(t, T, $\Omega^2{}_t$), such that $\Omega^1{}_t \subseteq \Omega^2{}_t$, it is said that g(.) is their stock differential rate if it fulfills:

$$< 1 + r(t, T, \Omega^1{}_t) > . < 1 + g(t, T, \Omega^1{}_t \rightarrow \Omega^2{}_t) > \; = \; < 1 + r(t, T, \Omega^2{}_t) >$$

Definition 2 <u>*Flow Differential Rate*</u>

Given r(t, T, $\Omega^1{}_t$) and r(t, T, $\Omega^2{}_T$), such that $\Omega^1{}_t \subseteq \Omega^2{}_T$, it is said that g(.) is their flow differential rate if it fulfills:

$$< 1 + r(t, T, \Omega^1{}_t) > . < 1 + g(t, T, \Omega^1{}_t \rightarrow \Omega^2{}_T) > \; = \; < 1 + r(t, T, \Omega^2{}_T) >$$

02.01 NUMERICAL EXAMPLE: Stock differential rates

<u>Setting:</u>

An investor wants to assess the impact of certain transaction costs in buying a financial asset.

<u>Data:</u>

Transaction costs amounts to 1,5 % of the asset buying price. The security grants a nominal annual return of 9 % for a holding period of six months.

<u>Valuation:</u> By definition 1

$$\mathbf{< 1 + r(t, T, \Omega^1{}_t) > . < 1 + g(t, T, \Omega^1{}_t \rightarrow \Omega^2{}_t) > \; = \; < 1 + r(t, T, \Omega^2{}_t) >}$$

and we make:

$$\mathbf{r(t, T, \Omega^1{}_t) = 0.0150 \qquad r(t, T, \Omega^2{}_t) = 0.0450}$$

hence, the differential rate yields:

$$\mathbf{g(t, T, \Omega^1{}_t \rightarrow \Omega^2{}_t) = 0.0296}$$

Appraisal:

What does **g(,)** mean?

Firstly, that the rate of return excluding the cost of transaction reaches only 2,96 % on the holding period. Secondly, it would be the maximum return the investor could claim unless there were other significant transaction costs to take into account.

Rounding Off:

If the investor had committed 5 million dollars to his investment, he would have earned a nominal return of 225,000 dollars. (0.045 x 5 million). However, he would have earned 148,000 dollars as net return (0.0296 x 5 million), and the difference of 77,000 dollars (225,000 – 148,000) would have been allocated as transaction costs.

We have to bear in mind, nevertheless, that we have rounded off figures in the paragraph above. In fact, with the four-digits format for the net rate of return, we would get:

$$5{,}000{,}000 \times (1 + 0.0296) = 5{,}148{,}000 \text{ dollars}$$

$$5{,}148{,}000 \times (1 + 0.0150) = 5{,}225{,}220 \text{ dollars}$$

But with the nominal return the final amount would be : 5,225,000 dollars.

By using a six-digits format for the rate of return net of transaction costs, we would have got the value g(.) = 0.029557 (which it comes out of : 1.045000 / 1.015000 = 1, 029557) .Plugging in this rate, the total amount would have been 5,000,000.00002. Working with this longer precision, the allocations would have been: 147,783.25 as net returns, and 77,216.75 as transacction costs.

02.02 NUMERICAL EXAMPLE: Flow differential rates

Setting:

A company Treasurer budgets a rate of interest for two months ahead as 7 % (annual).

Data:

Two months later, at the time of depositing the money, banks pay no more than 6 % (annual).

Valuation:

By definition 2,

$$< 1 + r(t , T , \Omega^1{}_t) > . < 1 + g(t , T , \Omega^1{}_t \rightarrow \Omega^2{}_T) > = < 1 + r(t , T , \Omega^2{}_T) >$$

if we suppose a year-basis of 360 days:

$$\mathbf{r(t , T , \Omega^{1}{}_{t}) = 0.07 / 12 = 0.0058}$$

$$\mathbf{r(t , T , \Omega^{2}{}_{T}) = 0.06 / 12 = 0.0050}$$

hence, the differential rate which measures the budget gap yields:

$$\mathbf{g(t , T , \Omega^{1}{}_{t} \rightarrow \Omega^{2}{}_{T}) = - 0.0008}$$

Appraisal:

If the Treasurer had wanted to deposit 10 million dollars, for instance, the expected interest income to earn would have amounted to 58,000 dollars. But the real outcome would have been only 50,000 dollars. The gap of 8,000 dollars comes out of

$$g(.) \ x \ 10{,}000{,}000 = - 80{,}000 \text{ dollars}$$

Rounding Off: it holds true what it was said in 02.01.

03.- THE REVERSE DIFFERENTIAL RATE

Let us work with stock differential rates, as they are the main concern in this paper. However, extension to flow differential rates runs outright. By definition 1,

$$\mathbf{< 1 + r(t , T , \Omega^{1}{}_{t}) > . < 1 + g(t , T , \Omega^{1}{}_{t} \rightarrow \Omega^{2}{}_{t}) > = < 1 + r(t , T , \Omega^{2}{}_{t}) >}$$

What if we wanted to know how to explain $\mathbf{r(t , T , \Omega^{1}{}_{t})}$ from the "global " rate $\mathbf{r(t , T , \Omega^{2}{}_{t})}$? We would have to discount the latter to reach the former. This task is accomplished by the reverse differential rate.

$$\mathbf{b(t , T , \Omega^{2}{}_{t} \rightarrow \Omega^{1}{}_{t})}$$

and it leads to the equation

[03]

$$\mathbf{< 1 + r(t , T , \Omega^{1}{}_{t}) > = < 1 + r(t , T , \Omega^{2}{}_{t}) > . < 1 + b(t , T , \Omega^{2}{}_{t} \rightarrow \Omega^{1}{}_{t}) >}$$

By the same token, whenever we deal with flow differential rates, the reverse comes out of:

$$\mathbf{< 1 + r(t , T , \Omega^{1}{}_{t}) > = < 1 + r(t , T , \Omega^{2}{}_{T}) > . < 1 + b(t , T , \Omega^{2}{}_{T} \rightarrow \Omega^{1}{}_{t}) >}$$

The following lemma show how strong is the relationship between any differential rate and its reverse. We proceed along stock differential rates, bearing in mind the minor changes we should follow each time flow differential rates were involved.

Lemma 1: ***Differential rates g(.) and their corresponding reverse rates b(.) fulfill the following relation***

[04]

$$< 1 + g(t, T, \Omega^1{}_t \rightarrow \Omega^2{}_t) > . < 1 + b(t, T, \Omega^2{}_t \rightarrow \Omega^1{}_t) > = 1$$

Proof: multiplying [03] by $< 1 + g(t, T, \Omega^1{}_t \rightarrow \Omega^2{}_t) >$, we get [04] outright. χ

Appraisal: It is worth expanding on three consequences this lemma provides:

a) Relation [04] translates an equivalence between both gap measures. In fact, **g(,)** performs as if were an accrued rate of interest, and **b(,)** as if were a discount rate of interest, in the framework of an interest rates arbitrage taking place in the money market:

$$[1 + i(t, T)] . [1 - d(t, T)] = 1$$

b) The difference between these financial mathematic relations lies in that any of these differential gaps, **g(,)** and **b(,),** can adopt negative signs, remaining positive the other one. Instead, the discount rate is the only to be preceded by the negative sign in a), just because **i(t,T)** is a nominal rate of interest which is always greater than zero.

c) Last of all, Lemma 1 allows to net a gross rate or, symmetrically, to gross up a net rate, by means of differential rate.

03.01 NUMERICAL EXAMPLE: Reverse differential rates

Setting:

Going on with the example 02.01, we can ask for the rate of discount the nominal rate of return will suffer to produce the 2,96 % at last.

Valuation: By [03]

$$< 1 + r(t, T, \Omega^1{}_t) > = < 1 + r(t, T, \Omega^2{}_t) > . < 1 + b(t, T, \Omega^2{}_t \rightarrow \Omega^1{}_t) >$$

that is to say:

$$1{,}0296 = 1{,}045 . < 1 + b(t, T, \Omega^2{}_t \rightarrow \Omega^1{}_t) >$$

and hence

$$b(t, T, \Omega^2{}_t \rightarrow \Omega^1{}_t) = -0.0147$$

Appraisal:

Sometimes is more useful to translate the gap owed to transaction costs into a gross rate of return discount perspective and here, the reverse rate of return comes in handy.

04.- THE TRANSACTION COSTS FUNCTION

AS A MULTIPLICATIVE MODEL OF DIFFERENTIAL RATES

In order that any buying, short-selling, holding, or selling transaction of a financial asset might be rounded off, economic agents incur a wide structure of costs associated with the whole transaction along the horizon **[t ; T]** .

The transaction costs fundtion can be regarded as a multiplicative model of five differential rates (see Apreda, 2000-a), which give account for, namely, intermediation, taxes, information, microstructure and financial costs. By the same token, each of those differential rates could be broken down into as many specific transaction costs as the analyst regards worthwhile.

Remark on notation:

TC, INT, MICR, FIN, TAX, INF must be read as the corresponding cost rates. Namely: transaction costs, intermediation costs, microstructure costs, financial costs associated with the transactions, tax structure, and information costs.

Definition 3 *<u>The Transaction Costs Function</u>*

$$< 1 + TC\,(t, T, \Omega^{TC}{}_t) > \; = \; < 1 + INT\,(t, T, \Omega^{INT}{}_t) > . < 1 + MICR(t, T, \Omega^{MICR}{}_t) >$$

$$. < 1 + MICR\,(t, T, \Omega^{INT}{}_t) > . < 1 + INF\,(t, T, \Omega^{INT}{}_t) > . < 1 + FIN\,(t, T, \Omega^{INT}{}_t) >$$

with the restriction that

$$\Omega^{k}{}_t \subseteq \Omega^{TC}{}_t \quad \text{for } k: \text{ INT, MICR, INF, FIN, TAX}$$

04.01 NUMERICAL EXAMPLE : The transaction costs function as a complex of differential rates

<u>Setting:</u>

An investment fund manager would be ready to make an important buying order abroad to change his portfolio risk-return profile.

<u>Data:</u>

The analysis of the transaction costs function shows the following features:

a) Intermediation costs:	1.10 %
b) Taxes:	0.80 %
c) Information costs:	0.90 % (mainly tax, legal and trading advisory for both markets)
d) Microstructure:	1.20 % (regulations in both capital markets and foreign exchange trading procedures)

e) Financial costs 1.40 % (mainly for marginal account fees to mark to market the future contract, collaterals and a loan to complete the whole amount to invest)

Valuation:

By definition 3

$$< 1 + TC (t , T , \Omega^{TC}{}_{t}) > \; = \; < 1 + INT (t , T , \Omega^{INT}{}_{t}) > . < 1 + MICR(t , T , \Omega^{MICR}{}_{t}) >$$

$$. < 1 + MICR (t , T , \Omega^{INT}{}_{t}) > . < 1 + INF (t , T , \Omega^{INT}{}_{t}) > . < 1 + FIN (t , T , \Omega^{INT}{}_{t}) >$$

and replacing with data:

$$< 1 + TC (t , T , \Omega^{TC}{}_{t}) > \; = \; 1.0110 \;.\; 1.0080 \;.\; 1.0090 \;.\; 1.0120 \;.\; 1.0140$$

$$< 1 + TC (t , T , \Omega^{TC}{}_{t}) > \; = \; 1.0552$$

hence

$$TC (t , T , \Omega^{TC}{}_{t}) \; = \; 0.0552$$

Appraisal:

So, the investment manager faces an upfront cost of 5,52 % to be matched against the nominal return of the planned investment. Thus, he should think it over his decision portfolio from scratch, because other alternatives with fewer transaction costs could become more suitable for him.

05.- COMPUTING NET RATES OF RETURN FROM OF FINANCIAL ASSETS

As from now, our holding period will be **[t ; T].** The nominal total rate of return of buying, short-selling, holding and selling a financial asset along this period comes from the following relationship:

[05]

$$E[r (t, T, \Omega_{t})] \; = \; < E[P(T, \Omega_{t})] + E[I(t, T, \Omega_{t})] \; - \; P(t) \;] \; / \; P(t)$$

Remark on notation:

$E[r(t, T, \Omega_{t})]$: total expected nominal return provided by a financial asset in the holding period.

$E[P(T, \Omega_{t})]$: nominal expected selling price of the asset at valuation date " **T** ".

$P(t)$: nominal buying price of the asset at valuation date " t ".

$E[I(t, T, \Omega_{t})]$: any expected income to be collected from the asset during the holding period (for instance, dividends or coupon interests).

It is worth pointing out that, from the point of view of the analyst who tries to compute ex~ante and ex~post total holding returns from financial assets, **P(t)** can be either the actual buying price (the bid price quoted and settled by the dealer), or some sensible valuation model assessment, in all cases as if the analyst were about to buy the asset. By the same token, **P(T)** can be either the actual selling price at valuation date " **T** " (the asked price quoted and settled by the dealer), or by means of a future contract at valuation date " **t** " , or an ex~ante valuation model assessment, in all cases as if the analyst were about to sell the asset.

By regulations in their markets, It is usual that institutional investors ought to provide periodical information about many things, including returns in portfolios or single assets. In this case, the analyst follows a market value criterion and supposes he is about to sell the asset, in order to have the holding period return, although no selling will take place.

We proceed now to address two useful lemmas. The first tells us that the nominal return has two distinctive sources: a holding return and an income return. This very plain statement comes in handy to institutional investors and accountants. The second lemma comes as a powerful, although simple, statement, since it shows how to deal with net of transaction costs rates of return.

Lemma 2 : ***$E[\, r(t, T, \Omega_t) \,]$ can be broken down into holding returns and income returns.***

$$E[\, r(t, T, \Omega_t) \,] = E[\, hr(t, T, \Omega_t) \,] + E[\, ir(t, T, \Omega_t) \,]$$

Proof: it is enough to make

$$E[\, hr(t, T, \Omega_t) \,] = E[\, P(T, \Omega_t) \,] - P(t) \;/\; P(t)$$

$$E[\, ir(t, T, \Omega_t) \,] = E[\, I(t, T, \Omega_t) \,] \;/\; P(t) \;\chi$$

Now, if we attempt to introduce transaction costs in [05], we will have to bear in mind their actual sources. For ease of notation, we drop the information set Ω_t , as long as this discussion lasts.

a) **c(t)** , when buying the security, at valuation date " **t** ".

b) **c(T),** when selling the security, at valuation date " **T** ".

c) **c(t , T),** that accounts for all transactions costs incurred with the collection of any income cash flow composing **I(t , T).**

Therefore, we have to compute the net rate of return, during the holding period, with a cash-flow perspective:

a) Adding to the acquisition value **P(t)** the cost of the transaction, that gives **P(t) . c(t)** , we get:

$$P(t) \,.\, [\, 1 + c(t) \,]$$

That is to say, the investment we make is not given by the purchase value **P(t)** only, as is usually told in text-books, but by the expression above, because both items are outgoing cashflows .

b) By the same token, we substract from the selling value **P(T)** the cost of the transaction, that gives **P(T) . c(T)**

$$\mathbf{P(T).[1-c(T)]}$$

Here we have an incoming cash flow and an outgoing one, and the balance provide with a net cash flow.

c) Although the income component of [05] is treated as in point b),

$$\mathbf{I(t,T).[1-c(t,T)]}$$

we must give a further qualification. Perhaps the holding period is long enough to provide with more than one cash flow. In this case, each of them must be associated with their transaction costs, and the final expression would require a comprehensive transaction costs rate.

For example, suppose that two cash flows (carrying on dividends or coupons, for instance), $\mathbf{CF(t_1)}$ and $\mathbf{CF(t_2)}$, are collected along $\mathbf{[t;T]}$. Then, taking transaction costs into account would yield:

$$\mathbf{CF(t_1).[1-c(t_1)] + CF(t_2).[1-c(t_2)]}$$

This can also be translated as:

$$\mathbf{CF(t_1).[1-c(t_1)] + CF(t_2).[1-c(t_2)] = [CF(t_1) + CF(t_1)].[1-c(t,T)]}$$

Now, we can set up the following definition for a rate of return to become net of transaction costs.

Definition 4 <u>*Net Rate of Return*</u>

Conditional upon an information set Ω_t, the net rate of return net r(t,T), along [t ; T], comes from:

$$\mathbf{1 + net\ r(t,T) =}$$

$$\mathbf{\{P(T).[1-c(T)]\} / \{P(t).[1+c(t)]\} + \{I(t,T).[1-c(t,T)]\} / \{P(t).[1+c(t)]\}}$$

The following lemma holds true for any financial asset. Its proof was formerly given in a previous paper (Apreda 2000-a). For the ease of the reader, we have attached it in Appendix 1.

Lemma 3 : ***Conditional upon an information set Ω_t, the total return of a financial asset over the holding period [t ; T] can be expressed as a rate of return net of transaction costs. Besides, the total return can be broken down into a net return and a differential rate that accounts for transaction costs.***

$$1 + r(t,T) = [1 + net\ r(t,T)].[1 + g(t,T)]$$

05.01 NUMERICAL EXAMPLE : The net rate of return

Setting:

An investor wants to buy, hold for six months, and then sell certain financial asset. Here are his data:

Data:

- Transaction Costs:

$$c(t) = 1\ \% \qquad c(t,T) = 0.5\ \% \qquad c(T) = 0.8\ \%$$

- Expected Cash Flows:

$$P(t) = 76 \qquad E[\,P(T, \Omega_t)\,] = 82 \qquad E[\,I(t,T, \Omega_t)\,] = 5$$

Valuation:

He proceeds to compute the rates of return.

a) Nominal rate of return: by using [05]

$$r\,(t, T) = [\ P(T) + I(t, T) - P(t)\ ] \,/\, P(t)$$

$$r\,(t, T) = [\ 82 + 5 - 76\ ] \,/\, 76 = 0{,}1447$$

b) Then, he applies definition 4 to find out the net rate of return

$$1 + \text{net } r(t,T) =$$

$$\{\,P(T) \,.\, [1 - c(T)\,]\,\} \,/\, \{\,P(t) \,.\, [1 + c(t)]\,\} + \{\,I(t,T) \,.\, [1 - c(t,T)]\,\} \,/\, \{\,P(t) \,.\, [1 + c(t)]\,\}$$

and replacing by data:

$$1 + \text{net } r(t,T) = [\,82 \,.\, (\,1 - 0.008\,) - 5 \,.\, (\,1 - 0.0005) - 76 \,.\, (\,1 + 0.01)\,] \,/\, 76 \,.\, 1.01$$

$$1 + \text{net } r(t,T) = 1.1245$$

The net rate of returns is equal to 12,45 %.

c) At last, he needs to measure the " global " influence of transaction costs, by using

$$1 + r(\,t\,,T\,) = [\,1 + \text{net } r(\,t\,,T\,)\,] \,.\, [\,1 + g(\,t\,,T\,)\,]$$

replacing and solving, we get:

$$1.1447 = 1.1245 \,.\, [\,1 + g(\,t\,,T\,)\,] \quad \Rightarrow \quad g(\,t, T\,) = 0.0180$$

d) Alternatively, he could have used Lemma 3, by taking firstly

$$\alpha = P(T) \;/\; [\; P(T) + I(t,T) \;] = 0.9425$$

then

$$(1 + b(t,T)) = [1 - \alpha \,.\, c(T) - (1 - \alpha) \,.\, c(t,T)] \,.\, [1 \,/\, (1 + c(t))] \Rightarrow b(t,T) = -0.0177$$

and lastly, resoring to Lemma 1 so as to obtain **g(t, T) = 0.0180**. In appendix 1, the reader will find the proof of the lemma and the foundations for α.

06.- TIME DEPOSITS

A time deposit can be assimilated to a par bond with only one coupon. That can be seen from the statement of cash flows:

Gross acquisition price (money deposited) :	–	**P(t)**
Face Value (money returned):	+	**P(t)**
Coupon:	+	**P(t) . i(t , T)**

where **i(t , T)** is the actual rate of interest in the holding period.

Remark on the rate of interest:

In fact, the actual rate of return comes from doing:

$$i(t , T) = \text{Annual Nominal Rate} \;.\; (T - t) \;/\; \text{Year-basis}$$

If we attempted to measure the total return of this par bond, without transaction costs, as it is the usual although misleading procedure, we would get:

$$r (t, T) = [\; P(T) + I(t, T) - P(t) \;] \,/\, P(t)$$

but **P(T) = P(t)** because it is bought par; then:

$$r (t, T) = [\; P(t) + P(t) \,.\, i(t , T) - P(t) \;] \,/\, P(t)$$

therefore, by Lemma 2,

$$r (t, T) = i(t , T) = ir (t , T)$$

Let us introduce transaction costs and see how the picture changes outright:

$$\text{net } r (t, T) =$$

$$[\; P(T) \,.\, (1 - c(T)) + I(t, T) \,.\, (1 - c(t,T)) - P(t) \,.\, (1 + c(t)) \;] \,/\, P(t) \,.\, (1 + c(t))$$

and taking advantage that it is a par bond and we know how to make out the income component,

$$\textbf{net r (t, T)} = \mathbf{P(t) . [(1 - c(T)) + i(t, T) . (1 - c(t,T)) - (1 + c(t))] / P(t) . (1 + c(t))}$$

hence

$$\textbf{net r (t, T)} = \mathbf{[(1 - c(T)) + i(t, T) . (1 - c(t,T)) - (1 + c(t))] / (1 + c(t))}$$

and rearranging this last expression, it gives out

[07]

$$\textbf{net r (t, T)} = \mathbf{[i(t, T) . (1 - c(t,T)) - c(T)) - c(t)] / (1 + c(t))}$$

which tells us that in order for the return on a time deposit to be profitable the net income from interest must outweigh the transaction costs at the entrance and exit, everything discounted by the entrance cost.

What if both **c(t)** and **c(T)** had the same value?. Then, we would have to detract - 2 c(t) from **(1 – c(t,T))**.

Lemma 4: ***Conditional upon an information set Ω_t, the rate of return of a time deposit can be broken down into a net rate of return plus a differential rate that accounts for transaction costs.***

$$1 + r(t,T) = [1 + \text{net } r(t,T)] . [1 + g(t,T)]$$

Proof: from definition 4 and [07] we get

$$\mathbf{1 + \textbf{net r (t, T)} = [r(t, T) . (1 - c(t,T)) + (1 - c(T))] / (1 + c(t))}$$

and solving for **r(t, T)** :

$$\mathbf{r(t, T) = [(1 + c(t)) . (1 + \textbf{net r (t, T)}) - (1 - c(T))] / (1 - c(t, T))}$$

which is equivalent to

$$\mathbf{1 + r(t, T) = 1 + [(1 + c(t)) . (1 + \textbf{net r (t, T)}) - (1 - c(T))] / (1 - c(t, T))}$$

Hence:

$$\mathbf{1 + r(t, T) = [(1 + c(t)) . (1 + \textbf{net r (t, T)}) - (1 - c(T))] + (1 - c(t, T)] / (1 - c(t, T))}$$

and also:

$$\mathbf{1 + r(t, T) = < (1 + c(t)) . (1 + \textbf{net r (t, T)}) / (1 - c(t, T) > .}$$

$$\mathbf{. < 1 - [(1 - c(T)) / (1 + c(t)] + [(1 - c(t, T)) / (1 + c(t)] >}$$

which simplifies to:

$$\mathbf{1 + r(t, T) = (1 + net\ r(t, T)) . (1 + [(c(T) + c(t)) / (1 - c(t, T))]}$$

and we call:

$$\mathbf{1 + g(t, T) = (1 + [(c(T) + c(t)) / (1 - c(t, T))]}$$

06.01 NUMERICAL EXAMPLE : A time deposit with transaction costs

Setting:

An investor builds up a time deposit abroad.

Data:

At that moment, the nominal spot rate for 3 months deposits is 9 % (annual). The investor faces an income withholding tax of 1 % over gross income. Furthermore, entrance fees, inclusive of foreign currency trading, amounts to 0.5 %, whereas exit fees inclusive of foreign currency trading adds up to 0,8 %.

Valuation: Taking advantage of Lemma 4

$$\mathbf{1 + g(t, T) = (1 + [(c(T) + c(t)) / (1 - c(t, T))]}$$

replacing and solving for **g(.)** we get:

$$\mathbf{1 + g(t, T) = (1 + [(0.005 + 0.008) / (1 - 0.01)]}$$

$$\mathbf{g(t, T) = 0.013131}$$

now we make for the rate of return net of transaction costs, by means of

$$\mathbf{1 + r(t,T) = [1 + net\ r(t,T)] . [1 + g(t,T)]}$$

$$\mathbf{1 + 0.022500 = [1 + net\ r(t,T)] . [1 + 0.013131]}$$

and the rate of return net of transaction costs drops to:

$$\mathbf{net\ r(t, T) = 0.009248}$$

Appraisal:

The investor should set this investment against more profitable ones after transaction costs.

07.- ZERO COUPON BONDS

A zero-coupon bond yields only a holding return till maturity. That can be seen from the statement of cash flows:

Gross acquisition price (money deposited) : **–** **P(t)**

Face value (money returned): + **P(T)**

Income flows (coupon) : no income coupon, by definition

where **P(t)** < **P(T)**

If we attempted to measure the total return of this par bond, without transaction costs, as it is the usual although misleading procedure, we would get:

$$r\,(t, T) \;=\; [\; P(T) \;+\; I(t, T) \;-\; P(t)\;] \,/\, P(t)$$

but **I(t , T)** equals zero, then:

$$r\,(t, T) \;=\; [\; P(T) \;-\; P(t)\;] \,/\, P(t)$$

therefore, by Lemma 1:

$$r\,(t, T) \;=\; hr(\,t\,,T)$$

Let us introduce transaction costs and see how the picture changes outright:

$$\text{net } r\,(t, T) \;=\; [\, P(T) \,.\, (\,1 - c(T)\,) \;-\; P(t) \,.\, (\,1 + c(t)\,)\;] \,/\, P(t) \,.\, (\,1 + c(t)\,)$$

We can proceed to draw an interesting message from this last relationship:

$$\text{net } r\,(t, T) \;=\; [\, P(T) \;-\; P(t) \;-\; P(T) \,.\, c(T) \;-\; P(t) \,.\, c(t)\,)\;] \,/\, P(t) \,.\, (\,1 + c(t)\,)$$

which leads to

[08]

$$\text{net } r\,(t, T) \;=$$

$$[\, P(T) \;-\; P(t)\;] \,/\, P(t) \,.\, (\,1 + c(t)\,) \;-\; [\, P(T) \,.\, c(T) \;+\; P(t) \,.\, c(t)\;] \,/\, P(t) \,.\, (\,1 + c(t)\,)$$

and [08] means that the discounted holding return must outweigh the discounted related transaction costs so as to make a sensible net rate of return.

Lemma 5: ***Conditional upon an information set Ω_t, the rate of return for a zero-coupn bond can be broken down into a net rate of return plus a differential rate that accounts for transaction costs.***

Proof: If we apply definition 4, bearing in mind that I[t,T] = 0, it follows:

$$P(T) \,.\, (\,1 - c(T)\,) \;/\; P(t) \,.\, (\,1 + c(t)\,) \;=\; 1 \;+\; \text{net } r(\,t\,;T\,)$$

But zero coupon bonds fulfills

$$P(T)\ /\ P(t) = 1 + r(t\,;T)$$

thus:

$$[\,1 + r(t\,;T)\,]\ .\ (1 - c(T))\ /\ (1 + c(t)) = 1 + \text{net}\, r(t\,;T)$$

and making

[09]

$$1 + b(t\,;T) = (1 - c(T))\ /\ (1 + c(t))$$

it follows

$$[\,1 + r(t\,;T)\,]\ .\ [\,1 + b(t\,;T)\,] = 1 + \text{net}\, r(t\,;T)$$

and profitting from Lemma 1, it yields

[10]

$$1 + g(t\,;T) = (1 + c(t))\ /\ (1 - c(T))$$

and finally:

$$[\,1 + r(t\,;T)\,] = [\,1 + \text{net}\, r(t\,;T)\,]\ .\ [\,1 + g(t\,;T)\,]$$

07.01 NUMERICAL EXAMPLE : A Zero Coupon Bond with transaction costs

Setting:

An investment fund manager is ready to buy a zero coupon bond in a foreign exchange. But he is afraid of transaction costs.

Data:

Entrance costs amounts to 0.6 % of nominal, because the transaction involves buying foreign currency and dealer's spread. Exit costs adds to 0.4 % of nominal, because the investor will sell foreign currency and there will be a witholding tax.

Valuation:

Taking advantage of [10] in Lemma 5,

$$1 + g(t\,;T) = (1 + c(t))\ /\ (1 - c(T))$$

and solving

$$g(t\,;T) = 0.0100$$

Appraisal:

The analyst should carefully set this investment against others, mainly when taking into account taxes and inflation.

08.- FOREIGN EXCHANGE MARKET

We want to illustrate the case for transaction costs in a covered arbitrage of interest rates between a domestic and a foreign market, by which we could make a riskless profit. (An earlier attempt to handle this issue is to be found in Frankel-Levich (1973); for an unified approach to covered arbitrage of interest rates, see Apreda (1992)). Without loss of generality, let us suppose that it is in the foreign market where the chance arises. The arbitrage process entails the following stages:

a) At valuation date " **t** ", we sell a security in the domestic market (it could be a term deposit with maturity at " **T** ").

b) We buy foreign currency, at valuation date " **t** ".

c) We buy a security in the foreign currency, matching the same class of risk and maturity that the one we bought in the domestic market.

d) At maturity, we sell the security and sell the income in foreign currency to buy domestic currency.

It must hold, therefore, that:

[11]

$$[\, 1 + r_D\,(t, T)\,] \;<\; [\, 1 + r_F\,(t, T)\,] \,.\, [\, 1 + r_{SWAP}\,(t, T)\,]$$

Remark on notation:

$r_D\,(t, T)$: nominal and actual rate of interest, during the holding period, at the domestic money market.

$r_F\,(t, T)$: nominal and actual rate of interest, during the holding period, at the foreign money market.

$r_{SWAP}\,(t, T)$: nominal and actual rate of change of the foreign exchange rate in terms of domestic money, during the holding period. It is usually referred as "swap return".

We can give an equivalent expression to [11], in terms of the rates of exchange:

$$[\, 1 + r_D\,(t, T)\,] \;<\; [\, 1 + r_F\,(t, T)\,] \,.\, [\, P_{D/F}(T) \;/\; P_{D/F}(t)\,]$$

where $P_{D/F}(T)$ means the amount of domestic currency we receive by selling sell one unit of foreign currency, and $P_{D/F}(t)$ means the amount of domestic currency we need to purchase one unit of foreign currency. That is to say, they are the bid and the ask price, respectively. What if the arbitrage had been the other way, from the foreign market to the domestic one? In that case, $P_{D/F}(T)$ would have been the ask price, and $P_{D/F}(t)$ the bid price.

There are basically two sorts of transaction costs in arbitraging with currencies:

- Those associated with the foreign exchange markets, spot and forward
- Those associated with security markets, domestic and foreign

By lemma 3, when selling a security in the domestic market, the return can be broken down into a net rate of return, and a differential rate that stands for transaction costs:

[12]

$$[\,1 + r_D\,(t, T)\,] = [\,1 + \text{net } r_D\,(t, T)\,]\,.\,[\,1 + g_D\,(t, T)\,]$$

By the same token, when buying, holding and selling a security, amounts to:

[13]

$$[\,1 + r_F(t, T)\,] = [\,1 + \text{net } r_F\,(t, T)\,]\,.\,[\,1 + g_F\,(t, T)\,]$$

Last of all, buying foreign currency, holding it under the guise of a security, and selling it eventually, leads to a return decomposable in a net rate of return and a differential rate of transaction costs

[14]

$$[\,1 + r_{SWAP}\,(t, T)\,] = [\,1 + \text{net } r_{SWAP}\,(t, T)\,]\,.\,[\,1 + g_{SWAP}\,(t, T)\,]$$

Although [11] is customarily exhibited as the condition to lock-in an arbitrage against the domestic market, the introduction of transaction costs will always cast a doubt on the arbitrage feasibility.

Lemma 6 : ***Conditional upon an information set Ω_t, covered interest arbitrage against the domestic exchange in favour of the foreign exchange is granted, net of transaction costs, whenever the round-trip transaction costs gap g(t, T) fulfills:***

[15]

$$[\,1 + \text{net } r_D\,(t, T)\,] < [\,1 + \text{net } r_F\,(t, T)\,]\,.\,[\,1 + \text{net } r_{SWAP}\,(t, T)\,]\,.\,[\,1 + g(t, T)\,]$$

Proof: By using [12], [13] and [14] into [11], we get:

$$[\,1 + \text{net } r_D\,(t, T)\,]\,.\,[\,1 + g_D\,(t, T)\,]$$

$$< [\,1 + \text{net } r_F\,(t, T)\,]\,.\,[\,1 + g_F\,(t, T)\,]\,.\,[\,1 + \text{net } r_{SWAP}\,(t, T)\,]\,.\,[\,1 + g_{SWAP}\,(t, T)\,]$$

and calling

$$[\,1 + g(t, T)\,] = \{\,[\,1 + g_F\,(t, T)\,]\,.\,[\,1 + g_{SWAP}\,(t, T)\,]\,\} \;/\; [\,1 + g_D\,(t, T)\,]$$

relation [15] holds true.

08.01 NUMERICAL EXAMPLE : Covered Interest Arbitrage with Transaction Costs

<u>Setting:</u>

We are going to proceed further into this issue by giving an example on stages, where [29] holds true after transaction costs.

Data:

Let us suppose that the foreign market grants 12 % anual returns for a 90 days deposit, whereas the domestic market offers 16 % anual returns for a similarly risk-rated deposit with the same maturity. The spot rate of exchange amounts to 1,54 domestic unit to buy the foreign currency unit, whereas the future rate of exchange, at valuation date " t " shows a premiun of 100 basis points above the spot for selling foreign currency, quoted at 1,50.

Valuation:

a) Is there any arbitrage gap to take advantage of?

$$1 + r_D(t,T) = 1.0400$$

$$1 + r_F(t,T) = 1.025$$

$$1 + r_{SWAP}(t,T) = 1.6000 \ / \ 1.5400 = 1.0390$$

Checking out [29]:

$$[\,1 + r_D(t,T)\,] < [\,1 + r_F(t,T)\,] \,.\, [\,1 + r_{SWAP}(t,T)\,]$$

$$1.0400 < 1.0650$$

Therefore, an arbitrage could be set into motion.

b) Let us introduce transaction costs.

$c_D(t)$	$= 0.6\ \%$	$c_D(T)$	$= 0.8\ \%$	$c_D(t,T)$	$= 0.5\ \%$
$c_F(t)$	$= 0.5\ \%$	$c_F(T)$	$= 0.5\ \%$	$c_F(t,T)$	$= 0.5\ \%$
$c_{SWAP}(t)$	$= 0.8\ \%$	$c_{SWAP}(T)$	$= 1.0\ \%$	$c_{SWAP}(t,T)$	$=$ -

In order to to use [12], [13], [14] we have firstly to compute the transaction costs differential rate in each case.

- Domestic market:

Taking profit of what we know about financials assimilated to term deposits, by using [ ] and replacing the numerical data,

$$(1 + g_D(t,T)) = [\,(1 + c_D(t)) \,.\, (1 + i_D(t,T)\,] \ / \ [\,1 + i_D(t,T) \,.\, (1 - c_D(t,T)) - c_D(T)\,]$$

$$(1 + g_D(t,T)) = 1.0111$$

- Foreign market:

Taking profit of what we know about financials assimilated to term deposits, by using [ ] and replacing the numerical data,

$$(1 + g_F(t,T)) = [(1 + c_F(t)) . (1 + i_F(t,T)] / [1 + i_F(t,T) . (1 - c_F(t,T)) - c_F(T)]$$

$$(1 + g_F(t,T)) = 1.0100$$

For the foreign currency swap rate, we make use of what we know about zero-coupon bonds, by using [ ] and replacing the numerical data,

$$1 + g_{SWAP}(t ; T) = (1 + c(t)) / (1 - c(T))$$

$$1 + g_{SWAP}(t ; T) = 1.0181$$

c) Now, we proceed to compute the net rates of return for each case.

- Domestic market:

$$[1 + r_D(t, T)] = [1 + \text{net } r_D(t, T)] . [1 + g_D(t, T)]$$

$$[1 + \text{net } r_D(t, T)] = 1.0286 \quad \Rightarrow \quad \text{net } r_D(t, T) = 0.0286$$

- Foreign market:

$$[1 + r_F(t, T)] = [1 + \text{net } r_F(t, T)] . [1 + g_F(t, T)]$$

$$[1 + \text{net } r_F(t, T)] = 1.0149 \quad \Rightarrow \quad \text{net } r_F(t, T) = 0.0149$$

and, as regards the swap rate,

$$[1 + r_{SWAP}(t, T)] = [1 + \text{net } r_{SWAP}(t, T)] . [1 + g_{SWAP}(t, T)]$$

$$[1 + \text{net } r_{SWAP}(t, T)] = 1.0286 \quad \Rightarrow \quad \text{net } r_{SWAP}(t, T) = 0.0286$$

Appraisal:

Checking out [11] for the net rates of return:

$$[1 + \text{net } r_D(t, T)] < [1 + \text{net } r_F(t, T)] . [1 + \text{net } r_{SWAP}(t, T)]$$

$$1.0286 < 1.0357$$

the arbitrage would be meaningful, because transaction costs are fully met. But meaningful doesn't mean more profitable than alternative arbitrages.

09.- CONCLUSIONS

a) Differential and reverse differential rates have a say whenever we want to know about what remains of financial rates of returns after transaction costs.

b) The transaction costs function can be dealt with by means of a multiplicative model of different rates, each of them standing for distinctive transaction costs main categories: intermediation, information, microstructure, taxes and financial costs.

c) When we step down to real securities, the transaction costs approach proves not only handy but explanatory. Besides, it allows for neater information, about what financial assets are able to yield eventually.

d) Microstructure, trading and information should be given serious regard because they could have the last word when we attempt to pick out the real winners among market securities.

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APPENDIX 1: The proof of Lemma 3

Conditional upon an information set Ω_t , the total return of a financial asset over the holding period [t ; T] can be expressed as a rate of return net of transaction costs. Besides, the total return can be broken down into a net return and a differential rate that accounts for transaction costs.

Proof: We know, by definition 4, that

$$1 + \text{net } r(t,T) =$$

$$\{ P(T) \,.\, [1 - c(T)] \} \,/\, \{ P(t) \,.\, [1 + c(t)] \} + \{ I(t,T) \,.\, [1 - c(t,T)] \} \,/\, \{ P(t) \,.\, [1 + c(t)] \}$$

But the right hand can be translated in a convenient way as:

[A1]

$$\{ P(T) \,.\, [1 - c(T)] \} \,/\, \{ P(t) \,.\, [1 + c(t)] \} + \{ I(t,T) \,.\, [1 - c(t,T)] \} \,/\, \{ P(t) \cdot [1 + c(t)] \} =$$

$$[1 / (1 + c(t))] \;.\; [\; \{ P(T) + I(t,T) \} / \{ P(t) \} \; - \; \{ P(T)\, .c(T) + I(t,T)\, .\, c(t,T) \} / \{ P(t) \} \;] =$$

$$[1 / (1 + c(t))] \;.\; [\; \{ 1 + r(t,T) \} \; - \; \{ P(T)\, .\, c(T) \; + \; I(t,T)\, .\, c(t,T) \; \} / \{ P(t) \} \;]$$

Furthermore, it holds true that

[A2]

$$\{ P(T)\, .\, c(T) \; + \; I(t,T)\, .\, c(t,T) \; \} / \{ P(t) \} \;=$$

$$\alpha \;.\; [\{ P(T) + I(t,T) \} / \{ P(t) \}]\, .\, c(T) \; + (1 - \alpha)\, .\, [\{ P(T) + I(t,T) \} / \{ P(t) \}]\, .\, c(t,T)$$

whenever we take:

$$\alpha \;=\; P(T) \;/\; [\; P(T) \; + I(t,T) \;]$$

Now, we can see that [A2] is equivalent to:

[A3]

$$P(T)\, .\, c(T) \; + \; I(t,T)\, .\, c(t,T) \; \} / \{ P(t) \} \;=$$

$$\alpha \;.\; [\, 1 + \; r(t, 1T)\,]\, .\, c(T) \;\; + \; (1 - \alpha)\, .\, [\, 1 \; + \; r(t, T)\,]\, .\, c(t,T)$$

Therefore, replacing [A3] in [A1]

$$\{ P(T)\, .\, [\, 1 - c(T)\,] \} / \{ P(t)\, .\, [\, 1 + c(t)] \} \; + \; \{ I(t,T)\, .\, [1 - c(t,T)] \} / \{ P(t) \cdot [1 + c(t)] \} \;=$$

$$[1 / (1+ c(t))]\, .\, [\, \{ 1 + r(t, T) \} - \alpha\, .\, \{ 1 + r(t, T) \}\, .\, c(T) - (1 - \alpha)\, .\, \{ 1 + \; r(t, T) \}\, .\, c(t,T) \;] =$$

$$\{ 1 + r(t, T) \}\, .\, [\, 1 \; - \; \alpha\, .\; c(T) - \; (1 \; - \; \alpha)\, .\; c(t) \;]\, .\, [1 / (1 + c(t))]$$

Hence,

$$1 \; + \;\; \text{net}\; r\, (t,T) \;\;\; =$$

$$\{ 1 + r(t, T) \}\, .\, [\, 1 \; - \; \alpha\, .\, c(T) - \; (1 \; - \alpha)\, .\, c(t,T) \;]\, .\, [1 / \; (1 \; + c(t))\,]$$

and taking

$$(1 \; + \; b(t,T) \;) \;=\; [\, 1 - \alpha\, .\, c(T) - \; (1 - \alpha)\, .\, c(t,T) \;]\, .\, [1 / \; (1 + c(t))\,]$$

we get:

$$1 + \; \text{net}\; r(t\, , T) \quad = \quad [\, 1 \; + \; r(t\, , T)\,]\, .\, [\, 1 \; + \; b(t\, , T)\,]$$

By Lemma 1,

$$[\, 1 + \; g(t,T)\,]\, .\, [\, 1 + \; b(t,T)\,] \;=\; 1$$

and definition 4 can be rewritten in this form:

$$1 \; + \;\; r(t\, , T) \quad = \quad [\; 1 \; + \; \text{net}\; r(t\, , T)\,]\, .\, [\, 1 \; + \; g(t\, , T)\,] \qquad \blacksquare$$